Optimization and Multivariate Calculus

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These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Calculus

1.1 Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions, and $a, n \in \mathbb{R}$. Derivatives have following properties:

1. $(af)' = af'(x)$
2. $(f+g)' = f'(x) + g'(x)$
3. $(fg)' = f'g + fg'$
4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$

1.2 Antiderivatives

Integrals/Antiderivatives have the following properties:

1. $\int af(x)dx = a\int f(x)dx$
2. $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
3. $\int f(x)\, g(y)\, dx\, dy = \int f(x)dx \int g(y)dy$
4. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ where $a < c < b$
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
6. $\int_a^b cdx = c(b-a)$

1.3 Common Antiderivative

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where $n \neq 1$
2. $\int e^x dx = e^x + C$
3. $\int e^{f(x)} f'(x)dx = e^{f(x)} + C$
4. $\int \frac{1}{x} dx = ln(x) + C$
5. $\int \frac{1}{f(x)} f'(x)dx = ln(f(x)) + C$

1.4 Fundamental Theorem of Calculus

For numbers a and b, the definite integral of $f(x)$ from a to b is $F(b)-F(a)$ where $F(x)$ is an antiderivative of f:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F' = f$

1.5 Integration by Parts

We can use integration by parts to integrate some more complex expressions. The formula for integration by parts is:

$$\int u(x) \cdot v'(x)dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x)dx$$

Example

Using integration by parts, we can integrate the expression xe^{2x}:
Let $u(x) = x$, and $v'(x) = e^{2x}$. Thus $u'(x) = 1$ and $v(x) = \frac{1}{2}e^{2x}$. Using the integration by parts, we see that:

$$\begin{aligned} \int xe^{2x}dx &= x\frac{1}{2}e^{2x} - \int 1 \cdot \frac{1}{2}e^{2x}dx \\ &= \frac{1}{2}\left(xe^{2x} - \int e^{2x}dx\right) \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \end{aligned}$$

where $C \in \mathbb{R}$.

1.6 Chain Rule

Let $w = f(x, y)$ where f is a differentiable function of x and y. Let $x = g(t)$ and $y = h(t)$ where g and h are differentiable functions of t. Then by the chain rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Example

Let $w = x^3y^2 - x^2$ and $x = e^t$ and $y = cos(t)$.

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= \left(3x^2y^2 - 2x\right)\left(e^t\right) + \left(2x^3y\right)\left(-sin(t)\right) \\ &= \left(3e^{2t}cos^2(t) - 2e^t\right)\left(e^t\right) - \left(2e^{3t}cos(t)\right)\left(sin(t)\right) \end{aligned}$$