ASSIGNMENT 1

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(1)

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ -3 & 1 & 6 & 3 \\ 2 & -2 & -1 & -1 \end{array}\right]$$

RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\Rightarrow x=7 \quad y=6 \quad z=3$

(2)
$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 2 \\ -2 & 2 & 1 & 4 \\ 3 & 2 & 2 & 5 \\ -3 & 8 & 5 & 17 \end{array}\right]$$

RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad x_1 = 0 \qquad x_2 = 3/2 \qquad x_3 = 1$

(3) SINCE AB IS 2×2 AND B IS 2×2, THEN WE KNOW A IS 2×2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{matrix} 7a_{11} + 2a_{12} = 5 \\ 3a_{11} + a_{12} = 4 \end{matrix} \quad \Rightarrow \quad a_{11} = -3 \qquad a_{12} = 13$$

$$\begin{matrix} 7a_{21} + 2a_{22} = -2 \\ 3a_{21} + a_{22} = 3 \end{matrix} \quad \Rightarrow \quad a_{21} = -8 \qquad a_{22} = 27$$

$$A = \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$$

(4) a) $$D^2 f_{x,y} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 8y & 8x - 9y^2 \\ 8x - 9y^2 & -18xy \end{bmatrix}$$

b) $$D^2 f_{x,y} = \begin{bmatrix} 6y & 6x + (7/2)y^{-1/2} \\ 6x + (7/2)y^{-1/2} & \left(\frac{-7}{2}\right) x y^{-3/2} \end{bmatrix}$$

(5) $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix}$$

$$= 4 - 1 + 0$$

$$= 3$$

(6) $$X^{-1} = X^{-1}(X^T)^{-1}X^T$$

$$= (X^TX)^{-1}X^T$$

(7) NOTICE THAT THE MATRIX IS NOT ~~[illegible]~~ POSITIVE DEFINITE OR POSITIVE SEMIDEFINITE ~~[illegible]~~ AS THE FIRST ORDER LEADING PRINCIPAL MINOR (-1) IS LESS THAN 0.

1st ORDER LEADING PRINCIPAL MINOR

$\det(-1) = -1$

2^{nd} ORDER LEADING PRINCIPAL MINOR:

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0$$

NOTICE THAT THE MATRIX IS <u>NOT</u> NEGATIVE DEFINITE, SO NOW WE NEED TO CHECK ALL PRINCIPAL MINOR

1^{st} ORDER PRINCIPAL MINORS:

$$-1 \qquad -1 \qquad -2 \qquad \text{ALL} \leq 0 \quad \checkmark$$

2^{nd} ORDER PRINCIPAL MINORS

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0 \qquad \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 \qquad \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2$$

$$\text{ALL} \geq 0 \quad \checkmark$$

3^{rd} ORDER PRINCIPAL MINOR

$$\begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -2 + 2 = 0 \qquad \leq 0 \quad \checkmark$$

THE MATRIX IS <u>NEGATIVE</u> <u>SEMI-DEFINITE</u>.

8 $f = (x^2+2y^2+3z^2)e^{-(x^2+y^2+z^2)}$

FOCs

$\frac{\partial f}{\partial x}: 2x e^{-(x^2+y^2+z^2)} + x^2(-2x)e^{-(x^2+y^2+z^2)} + 2y^2(-2x)e^{-(x^2+y^2+z^2)}$

$+ 3z^2 e^{-(x^2+y^2+z^2)}(-2x) = 0$

$\Rightarrow (-2x)e^{-(x^2+y^2+z^2)}(x^2+2y^2+3z^2-1) = 0$

$\frac{\partial f}{\partial y}: (-2y)e^{-(x^2+y^2+z^2)}(x^2+2y^2+3z^2-2) = 0$

$\frac{\partial f}{\partial z}: (-2z)e^{-(x^2+y^2+z^2)}(x^2+2y^2+3z^2-3) = 0$

LET'S EXAMINE $\frac{\partial f}{\partial z} = 0$: THE FOC TELLS US THAT EITHER $-2z = 0$, $e^{-(x^2+y^2+z^2)} = 0$ OR $x^2+2y^2+3z^2-3=0$. WE CAN SAY SIMILAR THINGS ABOUT THE OTHER FOCs.

CRITICAL POINTS:	TYPE
$(0,0,0)$	LOCAL MIN
$(\pm 1, 0, 0)$	SADDLE POINTS
$(0, \pm 1, 0)$	SADDLE POINTS
$(0, 0, \pm 1)$	LOCAL MAXES

9) $\mathcal{L} = k x_1^{\alpha} x_2^{1-\alpha} - \lambda [p_1 x_1 + p_2 x_2 - I]$

a) KKT CONDITIONS

$$\frac{\partial \mathcal{L}}{\partial x_1} : \alpha k x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} : (1-\alpha) k x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0 \quad (2)$$

$$\lambda [p_1 x_1 + p_2 x_2 - I] = 0 \quad (3)$$

$$\lambda \geq 0$$

$$p_1 x_1 + p_2 x_2 \leq I \quad (4)$$

b) COMBINING (1) AND (2) YIELDS:

$$\frac{\alpha \cancel{k} x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) \cancel{k} x_1^{\alpha} x_2^{-\alpha}} = \frac{\cancel{\lambda} p_1}{\cancel{\lambda} p_2}$$

$$\Rightarrow \frac{\alpha}{(1-\alpha)} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

$$\Rightarrow \quad x_2 = \left(\frac{p_1}{p_2}\right) x_1 \left(\frac{1-\alpha}{\alpha}\right)$$

PLUG INTO THE BUDGET CONSTRAINT:

$$p_1 x_1 + \cancel{p_2} \left(\frac{p_1}{\cancel{p_2}}\right) x_1 \left(\frac{1-\alpha}{\alpha}\right) = I$$

$$\Rightarrow x_1^* = \frac{\alpha I}{p_1}$$

$$\Rightarrow x_2^* = \frac{(1-\alpha) I}{p_2}$$

10 $$\max_{x_1, x_2, \ldots, x_n} \; p \prod_{i=1}^{n} x_i^{\alpha_i} - \sum_{i=1}^{n} w_i x_i \qquad (1)$$

NOTICE $f(x_1, x_2, \ldots, x_n)$ IS CONCAVE, AND $-\sum_{i=1}^{n} w_i x_i$ IS CONCAVE, SO $p f(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i$ IS CONCAVE

WE NOW ONLY NEED TO LOOK AT FIRST ORDER CONDITIONS:

FOC

$$\frac{\partial f}{\partial x_i}: \quad p \frac{\alpha_i}{x_i} f(x_1, x_2, \ldots, x_n) = w_i \qquad \text{FOR } i = 1, 2, \ldots, n$$

DIVIDE $\partial f / \partial x_i$ BY $\partial f / \partial x_1$

$$\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_1}}: \quad \frac{\cancel{p} \frac{\alpha_i}{x_i} \cancel{f(x_1, x_2, \ldots, x_n)}}{\cancel{p} \frac{\alpha_1}{x_1} \cancel{f(x_1, x_2, \ldots, x_n)}} = \frac{w_i}{w_1}$$

$$\Rightarrow x_i^* = \frac{\alpha_i}{w_i} \frac{w_1}{\alpha_1} x_1^* \qquad \text{FOR } i = 1, 2, \ldots, n \qquad (2)$$

NOTICE IF WE PLUG IN (2) INTO (1):

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \left(\frac{\alpha_i}{w_i}\right)^{\alpha_i} \left(\frac{w_1}{\alpha_1}\right) x_1^*$$

NOW TAKING FOC ~~WRT~~ WITH RESPECT TO x_1 OF

$$p\, f(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i$$

YIELDS:

$$\frac{p}{\alpha_1} = \prod_{i=1}^{n} \left(\frac{w_i}{\alpha_i}\right)^{\alpha_i}$$

NOTICE, THIS ISN'T A FUNCTION OF x_i. THUS ANY $x_1 > 0$ IS A SOLUTION (AND WE CAN FIND ANY OTHER x_i WHERE $i = 2, 3, \ldots, n$ BY EQUATION (2).

(11) LET $u(x) = x^2$ AND $v'(x) = \sin x$

$\Rightarrow u'(x) = 2x$ AND $v(x) = -\cos x$

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx + C_1$$

$$\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx + C_1$$

WE HAVE TO USE INTEGRATION BY PARTS AGAIN

TO EVALUATE $\int 2x\cos x\,dx$:

$u(x) = 2x$ AND $v'(x) = \cos x$

$\Rightarrow u'(x) = 2$ AND $v(x) = \sin x$

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx + C_2$$

$$\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx + C_2$$

$$\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x\,dx + C$$

(12) $w = 2xy \qquad x = s^2 + t^2$ and $y = s/t$

$$\frac{dw}{ds} = \frac{dw}{dx}\frac{dx}{ds} + \frac{dw}{dy}\frac{dy}{ds}$$

$$= 2y(2s) + 2x\left(\frac{1}{t}\right)$$

$$= \frac{6s^2 + 2t^2}{t}$$

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dy}{dt}$$

$$= 2y(2t) + 2x\left(-s/t^2\right)$$

$$= 4s - \frac{2s(s^2+t^2)}{t^2}$$