Master's Mathcamp Final **Name:** ____________________

Summer 2019

Time Limit: 60 Minutes

Instructions: Some questions on this test may be a bit difficult. Relax, and answer all questions to the best of your ability (check every page to make sure you have answered everything). Note that partial solutions will receive partial credit, so putting something for a question will be better than leaving that question blank.

Here are some useful derivatives:

$$\frac{d\ln x}{dx} = \frac{1}{x}$$

$$\frac{d\ln f(x)}{dx} = \frac{f'(x)}{f(x)}$$

$$\frac{dx^n}{dx} = nx^{n-1}$$

$$\frac{df(x)^n}{dx} = nf'(x)f(x)^{n-1}$$

1. **(40 points)** Consider the following consumer utility max problem (and assume price $m \geq p \geq 1$):

$$\max_{x_1,x_2} \ln x_1 + x_2$$

such that

$$x_1 + px_2 \leq m$$

(a) Calculate the Hessian, $D^2 f_{(x_1,x_2)}$, of the objective function, $f(x_1,x_2) = \ln x_1 + x_2$, and show that it is negative semi-definite over the domain $x_1 \geq 0, x_2 \geq 0$ (and thus f is concave).

The domain should be $x_1 > 0$, $x_2 \geq 0$.

$$Df_{(x_1,x_2)} = \begin{bmatrix} \frac{1}{x_1} & 1 \end{bmatrix}$$

$$D^2 f_{(x_1,x_2)} = \begin{bmatrix} -\frac{1}{x_1^2} & 0 \\ 0 & 0 \end{bmatrix}$$

Notice that for $x_1 \geq 1$, it follows that $-\frac{1}{x_1^2} < 0$. The first order principal minors ($-\frac{1}{x_1^2}$ and 0) are nonpositive, and the second order principal minor (0) is nonnegative. Thus the Hessian is negative semi-definite. This tells us that the objective function is concave.

(b) Define the Lagrangian and find the Karush-Kuhn-Tucker conditions (you don't have to include the nonnegativity constraints).

$$L = \ln x_1 + x_2 - \lambda(x_1 + px_2 - m) - \mu(1 - x_1)$$

KKT conditions

$$\frac{\partial L}{\partial x_1} : \frac{1}{x_1^*} - \lambda^* + \mu^* = 0 \tag{1}$$
$$\frac{\partial L}{\partial x_2} : 1 - \lambda^* p = 0 \tag{2}$$
$$\lambda^*(x_1^* + px_2^* - m) = 0 \tag{3}$$
$$\mu^* x_1^* = 0 \tag{4}$$
$$\lambda^*, \mu^* \geq 0 \tag{5}$$
$$x_1^* + px_2^* \leq m \tag{6}$$
$$x_1^* \geq 1 \tag{7}$$
$$x_2^* \geq 0 \tag{8}$$

(c) The objection function is "strictly increasing" in both inputs (x_1 and x_2) meaning that either increasing x_1 or x_2 will increase one's utility. That means that the budget constraint ($x_1 + px_2 \leq m$) is binding, or in other words holds with equality ($x_1 + px_2 = m$). Give economic intuition why this is the case.

The objective function (utility function) is strictly increasing in both inputs, meaning that the consumer will gain more utility by having more of either good. Thus, the consumer will spend all of his income (as he is better off buying x_1 and x_2).

(d) Using the conditions in (b) and (c), find the maximizers, x_1^* and x_2^*.

From equation (2), we find that $\lambda^* = \frac{1}{p}$, and from equation (1), we find that $x_1^* = \frac{1}{\lambda}$, thus $x_1^* = p$. Plugging this into the budget constraint, we find that $x_2^* = \frac{m}{p} - 1$.

2. **(15 points)** Use integration by parts to evaluate the following integral:

$$\int x\sqrt{x+1}\,dx$$

Recall that the formula for integration for parts is $\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$. Set $u(x) = x$ and $v'(x) = \sqrt{x+1}$. Thus, $u'(x) = 1$ and $v(x) = \frac{2}{3}(x+1)^{\frac{3}{2}}$.

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \frac{2}{3}(x+1)^{\frac{3}{2}}x - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\,dx \\ &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C\end{aligned}$$

3. **(15 points)** Solve the following system of linear equations:

$$\begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0 \end{aligned}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 2 & -4 & 3 & 0 \end{array}\right)$$

The reduced row echelon form is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Thus $x^* - 1$, $y^* = -1$, $z^* = -2$.