WSU Economics PhD Mathcamp Notes

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These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Linear Algebra

1.1 System of Linear Equations

A linear equation is an equation that can be written in the following form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \tag{1}$$

where $x_1, x_2, ..., x_n$ are the variables, and $a_1, a_2, ..., a_n$ are the coefficients or parameters.

Examples of linear equations

$$\begin{aligned} 4x_1 + 5x_2 &= 2 \\ 5x_1 - x_2 + 3x_3 &= \sqrt{5} \\ x_1 &= 10 \\ 7x_1 + 4x_3 &= -6 \end{aligned}$$

A system of linear equations is just a collection of linear equations that come from the same set of variables. An example is shown below:

$$2x_1 + x_2 = 5 \tag{2}$$

$$-x_1 + x_2 = 2 \tag{3}$$

To solve this simple system of linear equations (and get a solution set) we could use substitution, elimination, or even graphing. There are three different possibilities for the solution set:

- One solution
- Infinite solutions
- No solutions

1.1.1 Substitution Method

To use the substitution method, we need to solve one of the equations for one of the variables. For example, we could solve equation (3) for x_1: $x_1 = x_2 - 2$, and plug it back into equation (2) and solve for x_2:

$$\begin{aligned} 2(x_2 - 2) + x_2 &= 5 \\ 3x_2 &= 9 \\ x_2 &= 3 \end{aligned}$$

To solve for x_1, we could plug x_2 into either equation (2) or (3). We will use equation (3) for this example.

$$\begin{aligned} -x_1 + (3) &= 2 \\ x_1 &= 1 \end{aligned}$$

Thus, the solution to this system is (1,3).

1.1.2 Elimination Method

We could also use elimination to solve this system. To do this, we could multiply equation (3) by -1 and add it to equation (2):

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ +(x_1 - x_2 &= -2) \\ \\ \rightarrow 3x_1 &= 3 \\ x_1 &= 1 \end{aligned}$$

Now, we can take the solution for x_1 and plug it into either equations (2) or (3). We will plug it into equation (2):

$$\begin{aligned} 2(1) + x_2 &= 5 \\ x_2 &= 3 \end{aligned}$$

Thus, the solution to this system is (1,3), as we found when using the substitution method.

1.2 Matrices

In the subsections above, we were able to solve a simple system of linear equations using the methods outlined. This was fairly straightforward and effective, however, this methods may not be appropriate for systems that are larger or more complex. This is where matrix notation comes in handy. Below is an example of a coefficient matrix for (2) and (3):

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

The first column represents the variable x_1 and the second column represents the variables x_2. The rows represent each equation, and the numbers are the coefficients. We could also use (2) and (3) to write an augmented matrix:

$$\begin{bmatrix} 2 & 1 & | & 5 \\ -1 & 1 & | & 2 \end{bmatrix}$$

The size of the matrix is described by how many rows and columns it has (in that order). Thus the augmented matrix above is a 2×3 matrix, meaning there are 2 rows and 3 columns.

Using matrix notation, we can use the following elementary row operations to find solutions to $Ax = b$ (where A is an $m \times n$ matrix, x is an $n \times 1$ vector of variables, and b is an $m \times 1$ vector of scalars):

- interchange two rows in a matrix
- multiply a row by a nonzero constant
- modify a row by adding it to another row

Applying any of these operations to a matrix will result in a new matrix that is "row equivalent" to the original. The goal of using these operations is to get the matrix into row echelon form or even reduced row echelon form. A matrix is in row echelon form if:

- Nonzero rows are above rows containing only zero
- The first nonzero number in a row (also called the leading coefficient or pivot) is to the right of the leading coefficient of the row above it

The matrix below is in row echelon form.

$$\begin{bmatrix} 2 & 1 & 1 & | & 5 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

A matrix is in reduced row echelon form if:

- It is in row echelon form
- Every leading coefficient is 1 and is the only nonzero entry in its column

Below is an example of a matrix in reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 0 & 1 & | & 12 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Note

There are an infinite amount of matrices that are in row echelon form that are row equivalent to a certain matrix. However, there only exists a unique reduced row echelon matrix for that matrix.

Recall that each column in a matrix represents a variable (except for the last column in an augmented matrix). Assuming a matrix is in reduced row echelon the jth variable (corresponding to the jth column) is a basic variable if the column contains a leading coefficient. Otherwise, it is a free or nonbasic variable. In the matrix above, we see that the 1st and 3rd variables are basic variables, and the 2nd is a free variable.

Example

See page 138 for walkthrough of reducing a matrix to its row reduced echelon form.

Practice

Put the following coefficient matrix into reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix}$$

Recall that the solution set for $Ax = b$ can either have no solution, one solution, or infinitely many solutions. When reducing an augmented matrix to reduced row echelon form, then:

no solution	reduced row echelon form of the augmented matrix contains a row of the form: 0 0 ... 0 1 where the leading coefficient is in the rightmost columns
infinite solutions	there exists free variables in the reduced row echelon form
one solution	there exists no free variables in the reduced row echelon form

The following augmented matrices (already in reduced row echelon form) are examples of systems with solution sets described above.

No solution

$$\begin{bmatrix} 1 & 0 & 0 & | & 8 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

Infinite solution

$$\begin{bmatrix} 1 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

One solution

$$\begin{bmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 6 \\ 0 & 0 & 1 & | & 7 \end{bmatrix}$$

Practice 1

Solve the following system of equations:

$$\begin{aligned} 2x_1 - 2x_2 - 4x_3 &= 10 \\ 3x_1 - 3x_2 - 6x_3 &= -3 \\ -2x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

Practice 2

Solve the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 5 \\ 4x_1 - 7x_2 + x_3 &= -1 \end{aligned}$$

Practice 3

Solve the following system of equations:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

1.3 Linear Dependence

A set of vectors is *linearly dependent* if one of the vectors in the set can be defined as a linear combination of the other vectors in the set. Thus, vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ are linearly dependent if and only if there exist scalars $c_1, c_2, \ldots, c_p$ not all zero such that:

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0 \tag{4}$$

Therefore, $v_1, v_2, \ldots, v_p$ are *linearly independent* if and only if $c_1 = c_2 = \ldots = c_p = 0$ is the only solution to $c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$ (this is often called the trivial solution).

Note

If $p > n$, and $\{v_1, v_2, \ldots, v_p\}$ is in $\mathbb{R}^n$, then $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent.

1.4 Span and Basis

If $\{v_1, v_2, ..., v_p\}$ is a set of vectors in $\mathbb{R}^n$, a linear combination of this set can be expressed in the following form:

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p \tag{5}$$

The span (usually written $\text{Span}\{v_1, v_1, \ldots, v_p\}$) is the set of *all* linear combinations of vectors from a set. The span will form a subset (which we will call S) of $\mathbb{R}^n$

$\{v_1, v_2, ..., v_p\}$ form a basis of S if:

1. $\{v_1, v_2, ..., v_p\}$ span S
2. $\{v_1, v_2, ..., v_p\}$ are linearly independent

Note

A basis of $\mathbb{R}^n$ contains n vectors.

Example

$$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$$

- $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$ is a set of vectors in $\mathbb{R}^2$
- $\begin{bmatrix}7\\2\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, and $\begin{bmatrix}2\\1\end{bmatrix}$, since $\begin{bmatrix}7\\2\end{bmatrix} = 7 \cdot \begin{bmatrix}1\\0\end{bmatrix} + 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + 0 \cdot \begin{bmatrix}2\\1\end{bmatrix}$
- The Span$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$ is $\mathbb{R}^2$ as all linear combinations of the set includes $\mathbb{R}_2$
- $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ form a basis for $\mathbb{R}^2$

1.5 Dimension

The number of vectors that make up the basis of a subspace S in $\mathbb{R}^n$ is called the dimension of S (denoted $\dim(S)$).

Example

n vectors make up the basis for $\mathbb{R}^n$, thus $\dim(\mathbb{R}^n) =$ n

1.6 Rank

The rank has a number of definitions. Here is a list of some (equivalent) definitions:

- Number of nonzero rows in row echelon form
- The dimension of a matrix's row and column space
- Number of pivots or leading coefficients in row echelon form
- Number of linearly independent columns
- The maximal order of a non-zero minor[1].

A matrix is said to have full rank if its rank is equal to its number of columns or number of rows. In other words, an $m \times n$ matrix is full rank if the rank of the matrix is equal to $\min(m, n)$.

[1] A minor of order k is the determinant of a submatrix of size $k \times k$ within a matrix.

Practice 1

What is the rank for the following matrix:

$$\begin{bmatrix} 5 & -10 \\ -2 & 4 \end{bmatrix}$$

Practice 2

What is the rank for the following matrix:

$$\begin{bmatrix} 3 & 5 & -6 \\ -2 & 0 & 4 \end{bmatrix}$$

Exercises

1. Consider the following system of linear equations:

$$\begin{aligned} -3x_1 - x_2 + 2x_3 &= 7 \\ 2x_2 - 2x_3 &= 8 \\ 6x_1 - 3x_2 + 6x_3 &= -9 \end{aligned}$$

   (a) Put the system of linear equations into a **augmented** matrix.
   (b) Find the reduced row echelon form of the **augmented** matrix.
   (c) What is the rank of the **coefficient** matrix?

2. Consider the following system of linear equations:

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

   (a) Put the system of linear equations into a **coefficient** matrix.
   (b) Find the reduced row echelon form of the **coefficient** matrix.
   (c) What is the dimension of the row space the **coefficient** matrix?

3. What does the rank of a matrix tell us?