WSU Economics PhD Mathcamp Notes

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1 Real Analysis

1.1 Convex Sets

A set A, in a real vector space V, is convex iff:

$$\lambda x_1 + (1-\lambda)x_2 \in A$$

for any $\lambda \in [0,1]$ and any $x_1, x_2 \in A$.

1.2 Fixed Points

Let $f : X \to X$. A point x^* is a **fixed point** of f iff $f(x^*) = x^*$. Notice that if we apply the function f on a fixed point x^* multiple times, we still get x^* as an output.

1.3 Brouwer's Fixed Point Theorem

Let $(\mathbb{R}^n, d)$ be a metric space. If $f : X \to X$ be a continuous function, where X is a compact and convex subset of $\mathbb{R}^n$, then there exists an $x^* \in X$ such that $f(x^*) = x^*$.

Practice

Using the Intermediate Value Theorem, prove the Brouwer's Fixed Point Theorem in the metric space $(\mathbb{R}, d_1)$

1.4 Contraction

We will use the notation $f^n(x)$ to mean apply the function n times on x. In other words, if $n = 3$, then $f^3(x) = f(f(f(x)))$.

Let (X, d) be a metric space. A function $f : X \to X$ is said to be a contraction iff there exists a $\lambda < 1$ such that:

$$d(f(x), f(x')) \leq \lambda \cdot d(x, x')$$

for any $x, x' \in X$.

Notice that when a function is a contraction, when we apply the function to two points, the distance between the images are closer than the distance between the preimages.

Example

Let $([\frac{1}{2}, 10], d_1)$ be a metric space, and let f be a function from the set $[\frac{1}{2}, 10]$ into itself. We see that the function as $f(x) = \sqrt{x}$ on this metric space is a contraction. Notice that no matter what x we start with (where $x \in [\frac{1}{2}, 10]$), $\lim_{n\to\infty} f^n(x) = 1$.

Starting with $x = \frac{1}{2}$:
$f(\frac{1}{2}) = 0.707107$
$f^2(\frac{1}{2}) = 0.84090$
$f^3(\frac{1}{2}) = 0.917004$
$\vdots$
$f^{20}(\frac{1}{2}) = 0.999999$

1.5 Contraction Mapping Theorem

Let (X, d) be a complete metric space, and $f : X \to X$ be contraction. It follows that there exists a fixed point x^* of f, and for any $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

2 Homogeneity

We say that a function $f(x_1, x_2, ..., x_n)$ is homogeneous of degree k (commonly referred to as HDk) if:

$$f(\alpha x_1, \alpha x_2, ..., \alpha x_n) = \alpha^k f(x_1, x_2, ..., x_n) \text{ for all } \mathbf{x} \text{ and all } \alpha > 0 \tag{1}$$

In Economics, when a production is homogeneous of degree 1, it is said to have constant returns to scale (CRS). If $k > 1$, the production function has increasing returns to scale, and if $k < 1$, the production function has decreasing returns to scale.

Example

Consider the function: $f(x, y) = 5x^2y^3 + 6x^6y^{-1}$. To determine if this function if homogeneous, we need to multiply each input by α:

$$\begin{aligned} f(\alpha x, \alpha y) &= 5(\alpha x)^2(\alpha y)^3 + 6(\alpha x)^6(\alpha y)^{-1} \\ &= \alpha^{2+3}5x^2y^3 + \alpha^{6-1}6x^6y^{-1} \\ &= \alpha^5(5x^2y^3 + 6x^6y^{-1}) \\ &= \alpha^5(f(x, y)) \end{aligned}$$

This function is homogeneous of degree 5 (HD5).

2.1 Euler's Theorem

If we take the derivative of both sides of equation (1) by x_i, we get the following:

$$\begin{aligned} \frac{\partial f(\alpha x_1, \alpha x_2, ..., \alpha x_n)}{\partial x_i} \cdot \alpha &= \alpha^k \frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_i} \\ \frac{\partial f(\alpha x_1, \alpha x_2, ..., \alpha x_n)}{\partial x_i} &= \alpha^{k-1} \frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_i} \end{aligned} \tag{2}$$

We can use the result from equation (2) to derive Euler's Thereom:

If f is a C^1, homogeneous of degree k function on $\mathbb{R}^n_+$, then it follows:

$$x_1 \frac{\partial f(x)}{\partial x_1} + x_2 \frac{\partial f(x)}{\partial x_2} + \ldots + x_n \frac{\partial f(x)}{\partial x_n} = kf(x)$$

3 Definiteness of Matrix

3.1 Quadratic Form

A function $Q : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form if it is a homogeneous polynomial of degree two. Thus, a quadratic form can be written as:

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \tag{3}$$

where $a_{ij} \in \mathbb{R}$

Notice that equation (4) can be written using vectors and matrices:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & (a_{12}+a_{21})/2 & \dots & (a_{1n}+a_{n1})/2 \\ (a_{12}+a_{21})/2 & a_{22} & \dots & (a_{2n}+a_{n2})/2 \\ \vdots & \vdots & \ddots & \vdots \\ (a_{1n}+a_{n1})/2 & (a_{2n}+a_{n2})/2 & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{4}$$

$$= \mathbf{x}^T A \mathbf{x} \tag{5}$$

Notice that the coefficient matrix, which we will call A, is square and symmetric. There is an infinite amount of coefficient matrices that would yield the same quadratic form, however, it is convenient to define A in such a way that it is symmetric.

Example

Let $Q(\mathbf{x}) = 2x_1^2 + 3x_1x_2$. If we put this in matrix form, we would get the following result:

$$Q(\mathbf{x}) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Practice

Express the quadratic form $Q(\mathbf{x}) = -3x_1^2 + 2x_1x_2 + 4x_1x_3 - 2x_2^2 + 5x_2x_3$ in matrix form.

3.2 Definiteness

Consider an $n \times n$ symmetric matrix A. A is:

positive definite	if $\mathbf{x}^T A\mathbf{x} > 0 \quad \forall \mathbf{x} \in \mathbb{R}^n - \mathbf{0}$
negative definite	if $\mathbf{x}^T A\mathbf{x} < 0 \quad \forall \mathbf{x} \in \mathbb{R}^n - \mathbf{0}$
positive semidefinite	if $\mathbf{x}^T A\mathbf{x} \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n$
negative semidefinite	if $\mathbf{x}^T A\mathbf{x} \leq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n$
indefinite	if $\mathbf{x}^T A\mathbf{x} > 0$ for some $x \in \mathbb{R}^n$, and $\mathbf{x}^T A\mathbf{x} < 0$ for some $x \in \mathbb{R}^n$

Table 1: Definiteness

3.3 Principal Minors

We can also evaluate the principal minors of A to determine the definiteness of A.

Let A be an $n \times n$ matrix. A kth order principal submatrix is $k \times k$ and is formed by deleting $n - k$ rows, and the same $n - k$ columns. Taking the determinant of a kth order principal submatrix yields a kth order principal minor.

The kth order leading principal submatrix of A, usually written as $|A_k|$, is the left most submatrix in A that is $k \times k$. The determinant of the kth order leading principal submatrix is called the kth order leading principal determinant.

Let A be an $n \times n$ matrix. Then,

- A is positive definite iff all of its leading principal minors are positive.
- A is negative definite iff leading principal minors alternate in sign, and the 1st order princiapl minor is negative.
- A is positive semidefinite iff every principal minor of is is nonnegative.
- A is negative semidefinite iff every principal minor of odd order is nonpositive, and every principal minors of even order is nonnegative.
- A is indefinite iff A does not have any of these patterns.

Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Now assume that there is a $\lambda \in (0, 1)$ such that:

$$|f(x) - f(x')| \leq \lambda |x - x'|$$

for all $x, x' \in \mathbb{R}$

Suppose we start with $y_1 \in \mathbb{R}$ and construct a sequence (y_n) by a applying the function f at each index to the previous element of the sequence. Thus our sequence would look like the following:

$$\begin{aligned}(y_n) &= (y_1,\ y_2,\quad y_3,\qquad y_4,\qquad\quad ...)\\ &= (y_1, f(y_1), f(f(y_1)), f(f(f(y_1))), ...)\end{aligned}$$

Or in other words, $y_{n+1} = f(y_n)$.
You may find the following property of infinite series useful:

$$\sum_{i=1}^{\infty} ar^i = a \sum_{i=1}^{\infty} r^i = a \left(\frac{1}{1-r}\right)$$

where $a \in$ and $r \in (0, 1)$. In other words, this infinite sum is less than the constant: $a\left(\frac{1}{1-r}\right)$.

(a) Show that the sequence (y_n) is a Cauchy sequence.

(b) Since (y_n) is a Cauchy sequence, we see that (y_n) is a convergent sequence, or in other words there is a limit point y such that $\lim_{n\to\infty} y_n = y$. Prove that y is a fixed point of f.

2. Determine the definiteness of the following symmetric matrix:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$