WSU Economics PhD Mathcamp Notes

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These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Linear Spaces

We ended last lecture talking about vector spaces.

Vector Space

A vector space is a collection of vectors which can either be added or scalar multiplied. A vector space is a non-empty set V that has the following properties (assuming $v, w, z \in V$):

1. $\mathbf{u} + \mathbf{v} \in V$
2. $c\mathbf{v} \in V$ and $c \in \mathbb{R}$
3. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
4. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
5. $a(b\mathbf{w}) = (ab)\mathbf{w}$ where $a, b \in \mathbb{R}$
6. $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
7. For every $\mathbf{v} \in V$, there exists a $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = 0$
8. $1\mathbf{v} = \mathbf{v}$
9. $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$ for all $c \in \mathbb{R}$
10. $(k + c)\mathbf{u} = k\mathbf{u} + c\mathbf{u}$ for all $k, c \in \mathbb{R}$

The vector space that we most commonly work with is $\mathbb{R}^n$.

Example

Let $V = \left\{ \begin{bmatrix} x \\ 3x \end{bmatrix} : x \in \mathbb{R} \right\}$. Show that V is vector space.
In order to show that V is a vector space, we need to show that the properties listed above are satisfied. Let $u, v, w \in V$, where $u = \begin{bmatrix} u \\ 3u \end{bmatrix}$, $v = \begin{bmatrix} v \\ 3v \end{bmatrix}$, $w = \begin{bmatrix} w \\ 3w \end{bmatrix}$

1. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} u+v \\ 3u+3v \end{bmatrix} \in V$ since $u+v, 3(u+v) \in \mathbb{R}$

2. $c\mathbf{v} = \begin{bmatrix} cu \\ 3cu \end{bmatrix} \in V$ since $cu, 3cu \in \mathbb{R}$

3. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} u+v \\ 3u+3v \end{bmatrix} = \begin{bmatrix} v+u \\ 3v+3u \end{bmatrix} = \mathbf{v}+\mathbf{u}$

4. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \begin{bmatrix} (u+v)+w \\ (3u+3v)+3w \end{bmatrix} = \begin{bmatrix} u+(v+w) \\ 3u+(3v+3w) \end{bmatrix} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

5. $a(b\mathbf{w}) = \begin{bmatrix} a(bw) \\ a(3bw) \end{bmatrix} = \begin{bmatrix} (ab)w \\ (ab)3w \end{bmatrix} = (ab)\mathbf{w}$ where $a, b \in \mathbb{R}$

6. $\begin{bmatrix} 0 \\ 3 \times 0 \end{bmatrix} = \mathbf{0} \in V$

7. $\begin{bmatrix} v \\ 3v \end{bmatrix} + \begin{bmatrix} -v \\ -3v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

8. $1\mathbf{v} = 1\begin{bmatrix} v \\ 3v \end{bmatrix} = \begin{bmatrix} v \\ 3v \end{bmatrix} = \mathbf{v}$

9. $c(\mathbf{v}+\mathbf{w}) = \begin{bmatrix} c(v+w) \\ c(3v+3w) \end{bmatrix} = \begin{bmatrix} cv+cw \\ 3cv+3cw \end{bmatrix} = c\mathbf{v}+c\mathbf{w}$ for all $c \in \mathbb{R}$

10. $(k+c)\mathbf{u} = \begin{bmatrix} (k+c)u \\ (k+c)3u \end{bmatrix} = \begin{bmatrix} ku+cu \\ 3ku+3cu \end{bmatrix} = k\mathbf{u}+c\mathbf{u}$ for all $k, c \in \mathbb{R}$

Subspace

A subset U of V is called a subspace of V if it is also a vector space. To check if U is a subspace, you only need to check that the following properties hold:

1. **Additive Identity:** $\mathbf{0} \in U$
2. **Closed under addition:** $u+v \in U$ if $u, v \in U$
3. **Closed under multiplication:** if $a \in \mathbb{R}$ and $u \in U$, then $au \in U$

Example

Let $V = \left\{ \begin{bmatrix} x \\ 3x \end{bmatrix} : x \in \mathbb{R} \right\}$. Show that V is subspace of $\mathbb{R}^2$.
All we need to show now is that this subset satisfies the following three properties.

1. $\begin{bmatrix} 0 \\ 3 \times 0 \end{bmatrix} = \mathbf{0} \in V$

2. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} u+v \\ 3u+3v \end{bmatrix} \in V$ since $u+v, 3(u+v) \in \mathbb{R}$

3. $c\mathbf{v} = \begin{bmatrix} au \\ 3au \end{bmatrix} \in V$ since $au, 3au \in \mathbb{R}$

Example

Let $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$. Show that V is NOT subspace of $\mathbb{R}^2$.
All we need to show now is that this subset does not satisfy one of the properties of subspaces.

1. For a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ let $x = y = 0 \Rightarrow xy = 0 \Rightarrow \mathbf{0} \in U$

2. Let $u = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Notice that $u, v \in U$. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin U$

3. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ such that $u_1 u_2 \geq 0$. Note that $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$, and $c^2 u_1 u_2 \geq 0$. Thus $c\mathbf{u} \in U$

We see that the second property (Closed under addition) is not satisfied. Thus U is not a subspace of R^2.

2 Linear Transformations

Let V and W be vector spaces. $T : V \to W$ is a linear transformation if:

1. $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
2. $T(c\mathbf{x}) = cT(\mathbf{x})$

where $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$. We can combine these two conditions into one:

1. $T(c_1\mathbf{x} + c_2\mathbf{y}) = c_1 T(\mathbf{x}) + c_2 T(\mathbf{y})$

where $c_1, c_2 \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$.

Example

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(x) = 4x$. T is a linear transformation as:

$$T(c_1\mathbf{x} + c_2\mathbf{y}) = 4(c_1\mathbf{x} + c_2\mathbf{y}) = 4c_1\mathbf{x} + 4c_2\mathbf{y} = c_1 T(\mathbf{x}) + c_2 T(\mathbf{y})$$

Example

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(x) = 3x + 1$. T is NOT a linear transformation as:

$$T(c_1\mathbf{x} + c_2\mathbf{y}) = 3(c_1\mathbf{x} + c_2\mathbf{y}) + 1 = 3c_1\mathbf{x} + 3c_2\mathbf{y} + 1 \neq (3c_1\mathbf{x} + 1) + (3c_2\mathbf{y} + 1)$$

Example

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(x) = Ax$ such that A is $m \times n$ matrix. T is a linear transformation as:

$$T(c_1\mathbf{x} + c_2\mathbf{y}) = A(c_1\mathbf{x} + c_2\mathbf{y}) = c_1A\mathbf{x} + c_2A\mathbf{y} = c_1T(\mathbf{x}) + c_2T(\mathbf{y})$$

3 Eigenvalues and Eigenvectors

λ is an *eigenvalue* of an $n \times n$ matrix A if $A - \lambda I$ is singular (recall that a matrix is singular iff its determinant is 0). The non-zero $n \times 1$ vector $\mathbf{v}$ is an *eigenvector* of A if $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$. For a given matrix, there is a max of n eigenvalues, and n eigenvectors.

Example

Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. To find the eigenvalues of A, we first need to define $A - I_n$:

$$A - I_n = \begin{bmatrix} 0 - \lambda & 1 \\ -2 & -3 - \lambda \end{bmatrix}$$
$$\det(A - I_n) = \lambda^2 + 3\lambda + 2$$

$\lambda^2 + 3\lambda + 2$ is known as the *characteristic polynomial.* Now set the characteristic polynomial equal to 0:

$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda + 2)(\lambda + 1) = 0$$
$$\lambda = -2, \quad \lambda = -1$$

We can now use the eigenvalues to find the eigenvectors. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
First, consider $\lambda = -1$:

$$(A - (-1)I_n)\mathbf{v} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{v}$$

Set $(A - (-1)I_n) = 0$:

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{v} = \mathbf{0}$$
$$\Rightarrow v_1 + v2 = 0$$
$$v_1 = -v_2$$

We find that $v_1 = -v_2$, thus the eigenvector corresponding to the eignvalue of $\lambda = -1$ is $\mathbf{v} = \begin{bmatrix} k \\ -k \end{bmatrix}$ where $k \in \mathbb{R}$.

Now, consider $\lambda = -2$:

$$(A - (-1)I_n)\mathbf{v} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{v}$$

Set $(A - (-1)I_n) = 0$:

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{v} = \mathbf{0}$$
$$\Rightarrow 2v_1 + v2 = 0$$
$$v_2 = -2v_1$$

We find that $v_2 = -2v_1$, thus the eigenvector corresponding to the eignvalue of $\lambda = -2$ is $\mathbf{v} = \begin{bmatrix} c \\ -2c \end{bmatrix}$ where $c \in \mathbb{R}$.

4 Inner Product

The *standard inner product* or *dot product* of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, is denoted as $\mathbf{x} \cdot \mathbf{y}$ and is defined as:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

Exercises

1. Find the (real) eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

2. A diagonal matrix is a square matrix that has only zero value entries on the off-diagonal. Show that the eigenvalues of a diagonal matrix are the values on the diagonal of that matrix.

3. The distance between two $n \times 1$ vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as:

$$dist(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

   Redefine this distance formula using the inner product.