WSU Economics PhD Mathcamp Notes

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These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Proofs

Before we discuss proofs, we need to introduce some terminology. An **axiom** is a true statement that is accepted without proof. A **theorem** is a true statement that can be proven. Oftentimes, the term theorem is only used when talking about statements that have some sort of significance or importance. A **corollary** is a result that can be derived or deduced from a previous result. A **lemma** is a result that is used to establish another result.

1.1 Direct Proof

A **direct proof**, sometimes called a constructive proof, is a method of proof that is used to show $P \Rightarrow Q$. In a direct proof, we assume P to be true, and through a number of statements make our way to a statement that shows that Q is true. In order to demonstrate how to go about doing a direct proof, I shall present a few properties about integers:

1. The negative or any integer is also an integer
2. The summation of any two integers results is an integer
3. The product of any two integers results in an integer
4. Any even number can be written in the form: $2k$, where $k \in \mathbb{Z}$
5. Any odd number can be written in the form: $2k+1$, where $k \in \mathbb{Z}$

Example 1 Let $n \in \mathbb{Z}$. If n is odd, then $5n+9$ is even.

Result 1 Assume n is odd.
Thus n can be written in the following form: $2k+1$ where $k \in \mathbb{Z}$.
This means that $5n+9 = 5(2k+1)+9 = 10k+14 = 2(5k+7)$.
Notice that since $(5k+7) \in \mathbb{Z}$, therefore $5n+9$ is even.

Example 2 Let $n \in \mathbb{Z}$. If n is even, then $-3n-5$ is odd.

Result 2 Assume n is even.
Thus n can be written in the following form: $2k$ where $k \in \mathbb{Z}$.
This means that $-3n - 5 = -3(2k) - 5 = -6k - 5 = -6k - 5 = -6k - 6 + 1 = 2(-3k - 3) + 1$.
Notice that since $(-3k - 3) \in \mathbb{Z}$, therefore $-3n - 5$ is odd.

1.2 Proof by Contrapositive

The **contrapositive** for an implication $P \Rightarrow Q$ is defined as $(\sim Q) \Rightarrow (\sim P)$. Notice that $(\sim Q) \Rightarrow (\sim P)$ is the logical equivalent of $P \Rightarrow Q$. Proofs by contrapositive are very similar to direct proofs. The only difference is that we start with $\sim Q$ and and through a number of statements make our way to a statement that shows that $\sim P$ is true. Proofs by contrapositive are often used when it is easier to work with $\sim Q$ then it is to work with P.

Example 1 Let $n \in \mathbb{Z}$. If $3n - 9$ is even, then n is odd

Result 1 Assume n is even.
Thus n can be written in the following form: $2k$ where $k \in \mathbb{Z}$
This means that $3n - 9 = 3(2k) - 9 = 6k - 9 = 2(3k - 5) + 1$
Since $(3k - 10) \in \mathbb{Z}$, it follows that $3n - 9$ is odd.

Example 2 Let $n \in \mathbb{Z}$. $9n - 5$ is even if and only if n is odd.

Result 2 When a biconditional is involved, we need to prove both directions. In other words, we need to prove that ($9x - 5$ is even) $\Rightarrow$ (n is odd), and (n is odd) $\Rightarrow$ ($9x - 5$ is even). We will first show that ($9x - 5$ is even) $\Rightarrow$ (n is odd).

$\Rightarrow$) ($9n - 5$ is even) $\Rightarrow$ (n is odd)
Assume n is even.
Thus n can be written in the following form: $2k$ where $k \in \mathbb{Z}$
This means that $9n - 5 = 9(2k) - 5 = 2(9k - 3) + 1$
Since $9k - 3 \in \mathbb{Z}$, it follows that $9n - 5$ is odd.

$\Leftarrow$) (n is odd) $\Rightarrow$ ($9n - 5$ is even)
Assume n is odd.
Thus n can be written in the following form: $2m$ where $m \in \mathbb{Z}$
This means that $9n - 5 = 9(2m + 1) - 5 = 2(9m - 2)$
Since $9m - 2 \in \mathbb{Z}$, it follows that $9n - 5$ is even.

1.3 Cases

Oftentimes, it is easier to break the domain of a premise into subsets. The prover then works through the proof for each subset or **case**. Notice that these subsets need to exhaust the domain, meaning every point or element in the domain needs to be covered by a case. The following are examples of cases that could be used for their domains.

Example 1: $\forall x \in \mathbb{R}$:

Case 1: $x > 0$

Case 2: $x < 0$

Case 3: $x = 0$

Example 2: $\forall n \in \mathbb{Z}$:

Case 1: n is odd

Case 2: n is even

Example 3: Let $m, n \in \mathbb{Z}$. If mn is odd, then m and n are odd.

Result Assume m or n are even. Then, either m is even and n is odd, n is even and m is odd, or both m and n are even.

Case 1: Assume m and n are even.
Thus $m = 2r$ and $n = 2s$ where $r, s \in \mathbb{Z}$.
Therefore $mn = 2r \cdot 2s = 4rs = 2(rs)$.
Since $rs \in \mathbb{Z}$, it follows that mn is even.
Case 2: Assume without loss of generality that m is even and n is odd.
Thus $m = 2t$ and $n = 2u + 1$ where $t, u \in \mathbb{Z}$.
Therefore $mn = 2t \cdot (2u + 1) = 4tu + 2t = 2(2tu + t)$.
Since $2tu + t \in \mathbb{Z}$, it follows that mn is even.

Notice that in the previous example, we had three cases: either m is even and n is odd, n is even and m is odd, or both m and n are even. However, we only walked through 2 cases: both m and n are even, and m is even and n is odd. In case 2, we used the phrase **without loss of generality**, because both the cases of m is even and n is odd, and n is even and m is odd are similar, and so the proof of one case will be sufficient to cover the two cases.

1.4 Proof by Contradiction

If we are trying to prove $P \Rightarrow Q$, we assume both P and $\sim Q$ are true, and then we try to deduce a contradiction $(R \wedge (\sim R)\ )$. We usually start the proof by saying "Assume to the contrary..." or "Assume by contradiction that..." followed by P and $\sim Q$.

Example 1 Let $n \in \mathbb{Z}$. If n is even, then $5n + 3$ is odd.

Result 1 Assume to the contrary that there exists an even integer n such that $5n + 3$ is even.
Since n is even, we can write $n = 2k$ where $k \in \mathbb{Z}$.
Thus, $5n + 3 = 5(2k) + 3 = 10k + 3 = 2(5k + 1) + 1$.
Since $(5k + 1) \in \mathbb{Z}$, then $5n + 3$ is odd, which is a contradiction.

Example 2 Show that 100 cannot be written as a sum of one odd integer and two even integers.

Result 2 Assume to the contrary that 100 can be written as a sum of one odd integer and two even integers.
Thus, $100 = (2k + 1) + (2m) + (2j)$ where $k, n, j \in \mathbb{Z}$.
$100 = (2k + 1) + (2m) + (2j) = 2(k + m + j) + 1$.
Since $k + m + j \in \mathbb{Z}$, we see that 100 is odd. This is a contradiction.

1.5 Counterexample

We have used direct proof, proof by contrapositive, and proof by contradiction to show that $P \Rightarrow Q$. However, it is not always the case that $P \Rightarrow Q$. If we can find an x in the domain of the premise P such that Q is false, then it is *not* the case that $P \Rightarrow Q$.

Example 1 Disprove the following statement:

$$\text{If } x \in \mathbb{Z}, \text{ then } \frac{x^2 + 2x}{x^2 - 3x} = \frac{x + 2}{x - 3}$$

Result To disprove the statement above, we only need to provide a counterexample in the domain of P, in this case, an $x \in \mathbb{R}$ where the expression does not hold.
Consider $x = 0$. We see that $\frac{x^2+2x}{x^2-3x}$ is undefined at $x = 0$, however, $\frac{x+2}{x-3} = -\frac{2}{3}$ when $x = 0$. Thus, $x = 0$ is a counterexample to the statement above.

1.6 Mathematical Induction

Let P(n) be a statement, where $n \in \mathbb{N}$. To prove by induction, we need to prove two things:

1. A base case (usual base case is $n = 1$)
2. The inductive step: $\forall k \in \mathbb{N}$, the implication: $P(k) \Rightarrow P(k + 1)$ is true.

Example 1 Show that the sum of the first n positive integers is $n(n + 1)/2$. Or in other words:

$$1 + 2 + 3 + 4 + ... + n = n(n + 1)/2$$

Result 1 Let $P(n) : 1 + 2 + 3 + 4 + ... + n = n(n + 1)/2$ where $n \in \mathbb{N}$

1. **Base case**: $P(1) : 1 = 1(1 + 1)/2 = 1$. Thus the base case is true.
2. **Inductive step**: Assume $P(k)$ is true, thus:

$$P(k) : 1 + 2 + 3 + 4 + ... + k = k(k + 1)/2$$

   Now we show that $P(k+1)$ is true, or that $1+2+3+4+...+k+(k+1) = (k+1)(k+2)/2$
   $1+2+3+4+...+k+(k+1) = k(k+1)/2+(k+1) = k(k+1)/2+2(k+1)/2 = (k+2)(k+1)/2$
   By induction, $P(n)$ is true for every integer n.

Exercises

1. Let $n \in \mathbb{Z}$. Prove that if n is even, then $7n - 9$ is odd.
2. Let $a, b, m \in \mathbb{Z}$. Prove that if $2a + 3b \geq 12m + 1$, then $a \geq 3m + 1$ or $b \geq 2m + 1$.
3. Let $a, b \in \mathbb{Z}$. Prove that if $a + b$ and ab are of the same parity (either both are even or both are odd), then a and b are even.
4. Disprove the following statement. If $x, y \in \mathbb{R}$, then $log(xy) = log(x) + log(y)$.