Real Analysis

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August 1, 2019

1 Real Analysis

Consider the metric space $(\mathbb{R}, d_1)$. A set $S \subseteq \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in S$. We call b an **upper bound** of S. A set $S \in \mathbb{R}$ is **bounded below** if there exists $c \in \mathbb{R}$ such that $a \geq c$ for all $c \in A$. We call C a **lower bound** of S.
The **supremum** s, or least upper bound, for a set $S \subseteq \mathbb{R}$ if:

1. s is an upper bound of S.
2. if d is any upper bound of S, then $s \leq d$.

The **infimum**, or greatest lower bound, for a set $S \subseteq \mathbb{R}$ is defined similarly.

Consider the metric space $(\mathbb{R}, d_1)$. The number m is the **maximum** of a set S if $m \in S$, and $m \geq a$ for all $a \in S$. The **minimum** of a set can be defined similarly.

Practice

Determine the supremum and infimum for each of the following sets in R. Also determine if the supremum and infimum are equal to the maximum and minimum respectively of each set:

1. $[0, 2]$
2. $(0, 2)$
3. $[0, 2] \cup \{3\}$

2 Sequences

A **sequence** is a function $x : \mathbb{N} \to X$, which is either written as (x_n) or $\{x_n\}$. In other words, to define a sequence, we take numbers from the set of natural numbers (starting from 1 and counting up), and using a function, assign them to elements from a set X. Since a sequence is a function, we can write the n^{th} element of a sequence as $x(n)$, however, it is more commonly written as x_n. Recall that order does not matter for sets. This is not the case with sequences. For example, $(1, 1, 2, ...) \neq (1, 2, 1, ...)$. You will also notice that elements in a set can be repeated.

2.1 Sequence Convergence

Let (X, d) be a metric space. A sequence (x_n) **converges** to x if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, it follows that $d(x_n, x) < \epsilon$.

If the sequence (x_n) converges to x, then we can either write $\lim x_n = x$ or $(x_n) \to x$.

Consider the sequence (x_n), where $x_n = \frac{1}{\sqrt{n}}$. Find what it converges to, and show via proof that it converges to that value.

Proof

Let ε be an arbitrary positive number. Choose a natural number such that $N > \frac{1}{\varepsilon^2}$. Now, verify that this choice of N has the desired property. Let $n \geq N$. Then:

$$n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon$$

$$\text{Thus } |x_n - 0| < \varepsilon$$

Template for $(x_n) \to x$ proof in metric space $(\mathbb{R}, d_1)$:

- "Let ε be arbitrary."
- Choose an $N \in \mathbb{N}$ (this may take some work).
- Show that your choice of N works.
- "Assume $n \geq N$."
- Now derive the inequality $|x_n - x| < \varepsilon$.

A sequence converges to at most one limit.

Proof

We want to show that a sequence converges to at most one limit. I will show that this is the case when $(\mathbb{R}, d_1)$ is the metric space.

Proof: Suppose to the contrary that a sequence converges to both x and x' where $x \neq x'$.
Since $x \neq x'$, we know that $d(x, x') > 0$.
Let $\varepsilon = d(x, x')/2$.
By definition of convergence, we see that since $x_n \to x$, $\exists N_1$ s.t. $d(x_n, x) < \varepsilon$.
Also, we see that since $x_n \to x'$, $\exists N_2$ s.t. $d(x_n, x') < \varepsilon$.
Now, let $N = \max\{N_1, N_2\}$. If $n > N$, then $|x_n - n| < \varepsilon$ and $|x_n - x'| < \varepsilon$.
By the triangle inequality: $|x - x'| \leq |x_n - x| + |x' - x_n| < |x - x'|$.
But, this is a contradiction since $|x - x'| \not< |x - x'|$.

A sequence is said to be bounded if $\exists M > 0$ s.t. $|x_n| < M \quad \forall n \in \mathbb{N}$.

Practice

Show (via proof) that:

1. $\lim \frac{2}{\sqrt{2n+4}} = 0$

   Let's do some scratch work before we actually prove anything. Notice that:

$$\left|\frac{2}{\sqrt{2n+4}} - 0\right| = \frac{2}{\sqrt{2n+4}}$$

   Thus:

$$\begin{aligned} &\frac{2}{\sqrt{2n+4}} < \varepsilon \\ \Rightarrow &\frac{2}{\varepsilon} < \sqrt{2n+4} \\ \Rightarrow &\frac{4}{\varepsilon^2} < 2n+4 \\ \Rightarrow &\frac{2}{\varepsilon^2} - 2 < n \end{aligned}$$

   Therefore, we should choose $N > \frac{2}{\varepsilon^2} - 2$. Notice that we could also choose $N > \frac{2}{\varepsilon^2}$ as $N > \frac{2}{\varepsilon^2} > \frac{2}{\varepsilon^2} - 2$.

   Proof: Let $\varepsilon > 0$ be arbitrary. Choose N such that $N > \frac{2}{\varepsilon^2}$. Let $n \geq N$. Then:

$$\left|\frac{2}{\sqrt{2n+4}} - 0\right| < \varepsilon$$

2. $\lim \frac{4n+1}{2n+4} = 2$
   Let's do some scratch work before we actually prove anything. Notice that:

$$\begin{aligned} \left|\frac{4n+1}{2n+4} - 2\right| &= \left|\frac{4n+1}{2n+4} - 2\frac{2n+4}{2n+4}\right| \\ &= \left|\frac{4n+1}{2n+4} - \frac{4n+8}{2n+4}\right| \\ &= \frac{7}{2n+4} \end{aligned}$$

   Thus:

$$\begin{aligned} &\frac{7}{2n+4} < \varepsilon \\ \Rightarrow &\frac{7}{\varepsilon} < 2n+4 \\ \Rightarrow &\frac{7}{2\varepsilon} - 2 < n \end{aligned}$$

   Therefore, we should choose $N > \frac{7}{2\varepsilon} - 2$. Notice that we could also choose $N > \frac{7}{2\varepsilon}$ as $N > \frac{7}{2\varepsilon} > \frac{7}{2\varepsilon} - 2$.

   Proof: Let $\varepsilon > 0$ be arbitrary. Choose N such that $N > \frac{7}{2\varepsilon}$. Let $n \geq N$. Then:

$$\left|\frac{4n+1}{2n+4} - 2\right| < \varepsilon$$

2.2 Squeeze Theorem

If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and $\lim x_n = l$, $\lim z_n = l$, then $\lim y_n = l$.

2.3 Subsequence

A subsequence is derived from a sequence (x_n) by only keeping a subset of the elements while keeping the order of the sequence.

Example

A subsequence of the sequence $(1, 2, 3, 4, 5, \ldots)$ is $(1, 4, 5, \ldots)$.

2.4 Cauchy Criterion

A sequence is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|x_n - x_m| < \varepsilon$.

Every convergent sequence is a Cauchy sequence. The converse (i.e. every Cauchy sequence is convergent) is only true for certain metric spaces.

Practice

Consider the metric space $(\mathbb{R}, d_1)$. Show that every convergent sequence is a Cauchy sequence.

Proof: Let (x_n) be a convergent sequence, thus $x_n \to x$. Thus, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, n \geq N$, it follows that:

$$|x_n - x| < \frac{\varepsilon}{2}$$

and

$$|x_m - x| < \frac{\varepsilon}{2}$$

Using the triangle inequality (property 3 from metric spaces), we see that:

$$\begin{aligned} |x_m - x_n| &= |x_m - x + x - x_n| \\ &\leq |x_n - x| + |x_m - x| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus $|x_m - x_n| < \varepsilon$ when $m, n \geq N$. Therefore (x_n) is a Cauchy sequence.

Practice

Show that $(\frac{1}{n})$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$.
Notice that since $(\mathbb{R}, |\cdot|)$ is a metric space, then the following inequality holds by the traingle inequality (property 3):

$$|x_m - x_n| \leq |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + ... + |x_{n-1} - x_n|$$

Now for our specific sequence, we see that:

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \left|\frac{1}{m} - \frac{1}{m+1}\right| + \left|\frac{1}{m+1} - \frac{1}{m+2}\right| + \dots + \left|\frac{1}{n-1} - \frac{1}{n}\right|$$

We can play with the inequality above to get a desirable result. Notice that $\left|\frac{1}{m} - \frac{1}{m+1}\right| = \left|\frac{-1}{m(m+1)}\right| = \frac{1}{m(m+1)}$. If $m < n$, then $\left|\frac{1}{m} - \frac{1}{m+1}\right| > \left|\frac{1}{n-1} - \frac{1}{n}\right|$:

$$\begin{aligned}
\left|\frac{1}{m} - \frac{1}{n}\right| &\leq \left|\frac{1}{m} - \frac{1}{m+1}\right| + \left|\frac{1}{m+1} - \frac{1}{m+2}\right| + \dots + \left|\frac{1}{n-1} - \frac{1}{n}\right| \\
&< \left|\frac{1}{m} - \frac{1}{m+1}\right| + \left|\frac{1}{m} - \frac{1}{m+1}\right| + \dots + \left|\frac{1}{m} - \frac{1}{m+1}\right| \\
&< (m-n)\left|\frac{1}{m} - \frac{1}{m+1}\right| \\
&< m\left|\frac{1}{m} - \frac{1}{m+1}\right| \\
&= m\frac{1}{m(m+1)} \\
&= \frac{1}{(m+1)}
\end{aligned}$$

We want to show that for every $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $m, n \geq N$ where:

$$\left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon$$

If we set $\frac{1}{N+1} = \varepsilon \Rightarrow N = \frac{1}{\varepsilon} - 1$. Now we can use a more "formal" proof:

Let $\varepsilon > 0$. Choose N such that $N > \frac{1}{\varepsilon} - 1$. Then, for $m, n \in \mathbb{N}$ such that $m, n \geq N$, then:

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \varepsilon$$

Exercises

These exercises (especially 2 and 3) are a bit harder than the ones you have seen before. You will probably spend the majority of your homework time on these exercises. If you don't understand what to do, please discuss with others approaches that you could take. I could care less if you write down a proof perfectly, I mostly care that you can express your thoughts intuitively and can work through a problem. So even if you can't solve the proof, put your thoughts down so I can evaluate your thought process.

Consider the $(\mathbb{R}, d_1)$ metric space.

1. Show that $\lim \frac{4n+5}{5n+2} = \frac{4}{5}$.
2. Prove that a Cauchy sequence in this metric space is bounded.
3. Prove the squeeze theorem. Or in other words, show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n$, then $\lim y_n = l$.