Cauchy Sequence Proof

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Show that $(\frac{1}{n})$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$.
Notice that since $(\mathbb{R}, |\cdot|)$ is a metric space, then the following inequality holds by the traingle inequality (property 3):

$$|x_m - x_n| \leq |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + ... + |x_{n-1} - x_n|$$

Now for our specific sequence, we see that:

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \left|\frac{1}{m} - \frac{1}{m+1}\right| + \left|\frac{1}{m+1} - \frac{1}{m+2}\right| + \ldots + \left|\frac{1}{n-1} - \frac{1}{n}\right|$$

We can play with the inequality above to get a desirable result. Notice that $\left|\frac{1}{m} - \frac{1}{m+1}\right| = \left|\frac{-1}{m(m+1)}\right| = \frac{1}{m(m+1)}$. If $m < n$, then $\left|\frac{1}{m} - \frac{1}{m+1}\right| > \left|\frac{1}{n-1} - \frac{1}{n}\right|$:

$$\begin{aligned}
\left|\frac{1}{m} - \frac{1}{n}\right| &\leq \left|\frac{1}{m} - \frac{1}{m+1}\right| + \left|\frac{1}{m+1} - \frac{1}{m+2}\right| + \ldots + \left|\frac{1}{n-1} - \frac{1}{n}\right| \\
&< \left|\frac{1}{m} - \frac{1}{m+1}\right| + \left|\frac{1}{m} - \frac{1}{m+1}\right| + \ldots + \left|\frac{1}{m} - \frac{1}{m+1}\right| \\
&< (m-n)\left|\frac{1}{m} - \frac{1}{m+1}\right| \\
&< m\left|\frac{1}{m} - \frac{1}{m+1}\right| \\
&= m\frac{1}{m(m+1)} \\
&= \frac{1}{(m+1)}
\end{aligned}$$

We want to show that for every $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $m, n \geq N$ where:

$$\left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon$$

If we set $\frac{1}{N+1} = \varepsilon \Rightarrow N = \frac{1}{\varepsilon} - 1$. Now we can use a more "formal" proof:

Let $\varepsilon > 0$. Choose N such that $N > \frac{1}{\varepsilon} - 1$. Then, for $m, n \in \mathbb{N}$ such that $m, n \geq N$, then:

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \varepsilon$$