WSU Economics PhD Mathcamp Notes

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These notes are to accompany Mathematics for Economists by Simon and Blume. This is an early version of my notes. As such, please do not distribute them. Also, let me know if you find any errors.

1 Linear Algebra

1.1 System of Linear Equations

A linear equation is an equation that can be written in the following form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \tag{1}$$

where $x_1, x_2, ..., x_n$ are the variables, and $a_1, a_2, ..., a_n$ are the coefficients or parameters.

Examples of linear equations

$$\begin{aligned} 4x_1 + 5x_2 &= 2 \\ 5x_1 - x_2 + 3x_3 &= \sqrt{5} \\ x_1 &= 10 \\ 7x_1 + 4x_3 &= -6 \end{aligned}$$

A system of linear equations is just a collection of linear equations that come from the same set of variables. An example is shown below:

$$2x_1 + x_2 = 5 \tag{2}$$
$$-x_1 + x_2 = 2 \tag{3}$$

To solve this simple system of linear equations (and get a solution set) we could use substitution, elimination, or even graphing. There are three different possibilities for the solution set:

- One solution
- Infinite solutions
- No solutions

1.1.1 Substitution Method

To use the substitution method, we need to solve one of the equations for one of the variables. For example, we could solve equation (3) for x_1: $x_1 = x_2 - 2$, and plug it back into equation (2) and solve for x_2:

$$\begin{aligned} 2(x_2 - 2) + x_2 &= 5 \\ 3x_2 &= 9 \\ x_2 &= 3 \end{aligned}$$

To solve for x_1, we could plug x_2 into either equation (2) or (3). We will use equation (3) for this example.

$$\begin{aligned} -x_1 + (3) &= 2 \\ x_1 &= 1 \end{aligned}$$

Thus, the solution to this system is (1,3).

1.1.2 Elimination Method

We could also use elimination to solve this system. To do this, we could multiply equation (3) by -1 and add it to equation (2):

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ +(x_1 - x_2 &= -2) \\ \\ \rightarrow 3x_1 &= 3 \\ x_1 &= 1 \end{aligned}$$

Now, we can take the solution for x_1 and plug it into either equations (2) or (3). We will plug it into equation (2):

$$\begin{aligned} 2(1) + x_2 &= 5 \\ x_2 &= 3 \end{aligned}$$

Thus, the solution to this system is (1,3), as we found when using the substitution method.

1.2 Matrices

In the subsections above, we were able to solve a simple system of linear equations using the methods outlined. This was fairly straightforward and effective, however, this methods may not be appropriate for systems that are larger or more complex. This is where matrix notation comes in handy. Below is an example of a coefficient matrix for (2) and (3):

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

The first column represents the variable x_1 and the second column represents the variables x_2. The rows represent each equation, and the numbers are the coefficients. We could also use (2) and (3) to write an augmented matrix:

$$\begin{bmatrix} 2 & 1 & | & 5 \\ -1 & 1 & | & 2 \end{bmatrix}$$

The size of the matrix is described by how many rows and columns it has (in that order). Thus the augmented matrix above is a 2×3 matrix, meaning there are 2 rows and 3 columns.

Using matrix notation, we can use the following elementary row operations to find solutions to $Ax = b$ (where A is an $m \times n$ matrix, x is an $n \times 1$ vector of variables, and b is an $m \times 1$ vector of scalars):

- interchange two rows in a matrix
- multiply a row by a nonzero constant
- modify a row by adding it to another row

Applying any of these operations to a matrix will result in a new matrix that is "row equivalent" to the original. The goal of using these operations is to get the matrix into row echelon form or even reduced row echelon form. A matrix is in row echelon form if:

- Nonzero rows are above rows containing only zero
- The first nonzero number in a row (also called the leading coefficient or pivot) is to the right of the leading coefficient of the row above it

The matrix below is in row echelon form.

$$\begin{bmatrix} 2 & 1 & 1 & | & 5 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

A matrix is in reduced row echelon form if:

- It is in row echelon form
- Every leading coefficient is 1 and is the only nonzero entry in its column

Below is an example of a matrix in reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 0 & 1 & | & 12 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Note

There are an infinite amount of matrices that are in row echelon form that are row equivalent to a certain matrix. However, there only exists a unique reduced row echelon matrix for that matrix.

Recall that each column in a matrix represents a variable (except for the last column in an augmented matrix). Assuming a matrix is in reduced row echelon the jth variable (corresponding to the jth column) is a basic variable if the column contains a leading coefficient. Otherwise, it is a free or nonbasic variable. In the matrix above, we see that the 1st and 3rd variables are basic variables, and the 2nd is a free variable.

Example

See page 138 for walkthrough of reducing a matrix to its row reduced echelon form.

Practice

Problem 7.13.

Recall that the solution set for $Ax = b$ can either have no solution, one solution, or infinitely many solutions. When reducing an augmented matrix to reduced row echelon form, then:

no solution	reduced row echelon form of the augmented matrix contains a row of the form: 0 0 ... 0 1 where the leading coefficient is in the rightmost columns
infinite solutions	there exists free variables in the reduced row echelon form
one solution	there exists no free variables in the reduced row echelon form

The following augmented matrices (already in reduced row echelon form) are examples of systems with solution sets described above.

No solution

$$\begin{bmatrix} 1 & 0 & 0 & | & 8 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

Infinite solution

$$\begin{bmatrix} 1 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

One solution

$$\begin{bmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 6 \\ 0 & 0 & 1 & | & 7 \end{bmatrix}$$

Practice 1

Solve the following system of equations:

$$\begin{aligned} 2x_1 - 2x_2 - 4x_3 &= 10 \\ 3x_1 - 3x_2 - 6x_3 &= -3 \\ -2x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

Practice 2

Solve the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 5 \\ 4x_1 - 7x_2 + x_3 &= -1 \end{aligned}$$

Practice 3

Solve the following system of equations:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

1.3 Linear Dependence

A set of vectors is *linearly dependent* if one of the vectors in the set can be defined as a linear combination of the other vectors in the set. Thus, vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ are linearly dependent if and only if there exist scalars $c_1, c_2, \ldots, c_p$ not all zero such that:

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0 \tag{4}$$

Therefore, $v_1, v_2, \ldots, v_p$ are *linearly independent* if and only if $c_1 = c_2 = \ldots = c_p = 0$ is the only solution to $c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0$ (this is often called the trivial solution).

Note

If $p > n$, and $\{v_1, v_2, \ldots, v_p\}$ is in $\mathbb{R}^n$, then $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent.

1.4 Span and Basis

If $\{v_1, v_2, ..., v_p\}$ is a set of vectors in $\mathbb{R}^n$, a linear combination of this set can be expressed in the following form:

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p \tag{5}$$

The span (usually written $\text{Span}\{v_1, v_1, \ldots, v_p\}$) is the set of *all* linear combinations of vectors from a set. The span will form a subset (which we will call S) of $\mathbb{R}^n$

$\{v_1, v_2, ..., v_p\}$ form a basis of S if:

1. $\{v_1, v_2, ..., v_p\}$ span S
2. $\{v_1, v_2, ..., v_p\}$ are linearly independent

Note

A basis of $\mathbb{R}^n$ contains n vectors.

Example

$$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$$

- $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$ is a set of vectors in $\mathbb{R}^2$
- $\begin{bmatrix}7\\2\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, and $\begin{bmatrix}2\\1\end{bmatrix}$, since $\begin{bmatrix}7\\2\end{bmatrix} = 7 \cdot \begin{bmatrix}1\\0\end{bmatrix} + 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + 0 \cdot \begin{bmatrix}2\\1\end{bmatrix}$
- The Span$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$ is $\mathbb{R}^2$ as all linear combinations of the set includes $\mathbb{R}_2$
- $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ form a basis for $\mathbb{R}^2$

1.5 Dimension

The number of vectors that make up the basis of a subspace S in $\mathbb{R}^n$ is called the dimension of S (denoted $\dim(S)$).

Example

n vectors make up the basis for $\mathbb{R}^n$, thus $\dim(\mathbb{R}^n) =$ n

1.6 Rank

The rank has a number of definitions. Here is a list of some (equivalent) definitions:

- Number of nonzero rows in row echelon form
- The dimension of a matrix's row and column space
- Number of pivots or leading coefficients in row echelon form
- Number of linearly independent columns
- The maximal order of a non-zero minor[1].

A matrix is said to have full rank if its rank is equal to its number of columns or number of rows. In other words, an $m \times n$ matrix is full rank if the rank of the matrix is equal to $\min(m, n)$.

[1] A minor of order k is the determinant of a submatrix of size $k \times k$ within a matrix.

Practice

Problem 7.20.

2 Matrix Algebra

To add or subtract matrices together, the matrices must be of the same size. The results from these two operations will result in a matrix that is the same size of the matrices operated on. The addition or subtraction of matrices is done entrywise.

Addition

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{bmatrix} \tag{6}$$

Subtraction

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & \dots & a_{1n}-b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & \dots & a_{mn}-b_{mn} \end{bmatrix} \tag{7}$$

Scalar Multiplication

$$c \cdot \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & ... & ca_{mn} \end{bmatrix} \tag{8}$$

Matrix Multiplication

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{km} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \tag{9}$$

$$\begin{bmatrix} a_{11} \cdot b_{11} + ... + a_{1m} \cdot b_{m1} & \dots & a_{11} \cdot b_{1n} + ... + a_{1m} \cdot b_{mn} \\ \vdots & \ddots & \vdots \\ a_{k1} \cdot b_{11} + ... + a_{km} \cdot b_{m1} & \dots & a_{k1} \cdot b_{1n} + ... + a_{km} \cdot b_{mn} \end{bmatrix} \tag{10}$$

Matrix Multiplication Properties

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $c(AB) = (cA)B = A(rB)$
5. $A^k = A \cdot A \cdot \ldots \cdot A$

Transposes

When a matrix is transposed, the rows and columns are interchanged.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}^T = \tag{11}$$

$$\begin{bmatrix} a_{11} & \dots & a_{m1} \\ a_{12} & \dots & a_{m2} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix} \tag{12}$$

Transpose Properties

1. $(AB)^T = B^T A^T$
2. $(A^T)^T = A$
3. $(cA)^T = c(A)^T$
4. $(A + B)^T = A^T + B^T$
5. $(A^T)^{-1} = (A^{-1})^T$
6. $|A^T| = |A|$
7. If A has only real values, then $A^T A$ is positive-semidefinite

Advanced Practice

1. Show that $(AB)^T = B^T A^T \Rightarrow (ABC)^T = C^T B^T A^T$
2. Prove that if A has only real values, then $A^T A$ is positive-semidefinite

Trace

The trace of an $n \times n$ matrix, donated tr, is the sum of the (main) diagonal. If $A = \begin{bmatrix} 3 & 7 \\ 2 & 8 \end{bmatrix}$, then $tr(A) = 11$.

Determinants

It is a bit difficult to describe what a determinant is, but this discussion on stack exchange seems to give the most intuitive idea. A determinant can only be computed for a square matrix. The determinant for a matrix, A, can either be denoted as $|A|$ or $det(A)$.

The determinant of a scalar a is just a.

The determinant of a 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is:

$a_{11}a_{22} - a_{21}a_{12}$

The determinant of a 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is:

$-1^{1+1} \cdot a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + -1^{1+2} \cdot a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + -1^{1+3} \cdot a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Note

We don't have to use the first row to calculate the determinant of a matrix that's bigger than 2×2. For example, if I chose to use the 2nd column, the determinant for the matrix above would now be:

$-1^{1+2} \cdot a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + -1^{2+2} \cdot a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + -1^{3+2} \cdot a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

If the determinant of a square matrix is nonzero, then that matrix is nonsingular.

Properties

- $|A| = |A^T|$
- $|A||B| = |AB|$

Practice

Use the definition of a determinant for an $n \times n$ matrix to show that the determinant of a 2×2 matrix (which was defined earlier) is equal to $a_{11}a_{22} - a_{21}a_{12}$.

Inverses

An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix B such that:

$$AB = BA = I_n \tag{13}$$

where I_n is an $n \times n$ identity matrix (described in the special matrices section).

Inverse Properties

1. $(A^{-1})^{-1} = A$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(cA)^{-1} = c^{-1}A^{-1}$
4. If A, B, and C are invertible $n \times n$ matrices, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
5. $|A^{-1}| = |A|^{-1}$
6. $A^{-1}A = AA^{-1} = I$
7. $A^{-1} = \frac{1}{|A|}adj(A)$

The Invertible Matrix Theorem

The following properties for an $n \times n$ matrix A are equivalent (if one is true, all are true; if one is false, all are false):

- A is invertible
- A^T is invertible
- A has n leading coefficients
- There exists a matrix B such that $AB = I$
- There exists a matrix C such that $CA = I$
- The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.
- The equation $Ax = 0$ only has the trivial solution. In other words $x = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T$
- A is row equivalent to an $n \times n$ identity matrix
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- A is full rank

One way to find the inverse a matrix is to use the formula below:

$$A^{-1} = \frac{1}{|A|}adj(A)$$

If A is the following matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then:

$$C = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{21} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix} \tag{14}$$

And $adj(A) = C^T$.

Practice 1

Problem 8.22. Note: A^{-2} can also be written as $(A^{-1})^2$
Problem 9.2

Practice 2

Use the method outlined above to invert the following matrix:

$$\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

Practice 3

Use the method outlined above to invert the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{bmatrix}$$

Some Derivative of Matrices

The following examples show how to take derivatives when matrices are involved (I have only included 2 of the most relevant examples, I encourage you to further explore other properties). Let X be an $n \times k$ matrix, y be a $n \times 1$, and b be $k \times 1$.

1. $\frac{\partial b'X'Xb}{\partial b} = 2X'Xb$
2. $\frac{\partial b'X'y}{\partial b} = X'v$

Note

More advanced matrix techniques can be found at this link

2.0.1 Special Matrices

Square Matrix

The number of rows (n) equals the number of columns (n) for the matrix. The following is an example of a square matrix:

$$\begin{bmatrix} 10 & 5 & 9 \\ 4 & 4 & 3 \\ 6 & 17 & 2 \end{bmatrix} \tag{15}$$

Symmetric Matrix

A symmetric matrix has the following property: $A^T = A$. This means that $a_{ij} = a_{ji}$ for all i, j. Notice that this implies that a symmetric matrix has to be a square matrix ($n \times n$). The following is an example of a symmetric matrix:

$$\begin{bmatrix} 1 & 5 & 6 \\ 5 & 4 & 7 \\ 6 & 7 & 2 \end{bmatrix} \tag{16}$$

Idempotent Matrix

An idempotent matrix (A) has the following property: AA = A

Identity Matrix

An $n \times n$ identity matrix (either donated as I or I_n) has 1's on the diagonal and 0's elsewhere. Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Multiplying any matrix by an identity matrix will return that matrix.

$$AI = A \tag{18}$$

$$IB = B \tag{19}$$

Nonsingular Matrix

Another name for an invertible matrix is a nonsingular matrix. A nonsingular matrix has a nonzero determinant.

Orthogonal Matrix

A square matrix, Q, is orthogonal if:

$$Q^T Q = Q Q^T = I$$

Notice that this definition implies that $Q^T = Q^{-1}$.

Partition Matrix

A partitioned matrix is a matrix that is broken up into partitions (also called blocks).

$$\left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

In order to perform certain operations we need to have our partitioned matrices partitioned appropriately.

Addition and subtraction: If we are adding $A + B$ or subtracting $A - B$, we need them to be the same size. Also, they need to be partition the same way.

$$\left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$\left[\begin{array}{c|cc} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \hline b_{31} & b_{32} & b_{33} \end{array}\right] = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

Thus, $A + B$ will be defined as:

$$\left[\begin{array}{c|c} A_{11} + B_{11} & A_{12} + B_{12} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} \end{array}\right]$$

Matrix Multiplication: We can also matrix multiply two partitioned matrices. Notice that if we are multiply AB, the number of columns in A has to be equal to the number of rows in B. If this is satisfied, We can used partitioned matrices and treat the submatrices as elements. If X and Y are $m \times n$ matrices, and after partitioning they are defined as:

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$Y = \begin{bmatrix} E \\ F \end{bmatrix}$$

Then it follows that XY is defined as:

$$XY = \begin{bmatrix} AE + BF \\ CE + DF \end{bmatrix}$$

Notice that this implies that the number of columns of A has to be equal to the number of rows in E and F, and also the number of columns in B has to be equal to the number of rows in E and F.

3 Linear Spaces

Recall that $\mathbb{R}$ is the set of all real numbers. $\mathbb{R}^n$ where $n \geq 1$ is a set that contains all n-tuples of real numbers. In other words, a vector in $\mathbb{R}^n$ would contain n elements that are in R.

Note

A set of the form $\mathbb{R}^n$ is often referred to as an Euclidean space.

Vector Space

A vector space is a collection of vectors which can either be added or scalar multiplied. A vector space is a non-empty set V that has the following properties (assuming $v, w, z \in V$):

1. $u + v \in V$
2. $cv \in V$
3. $u + v = v + u$
4. $(u + v) + w = u + (v + w)$
5. $a(bw) = (ab)w$ where $a, b \in \mathbb{R}$
6. $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$
7. For every $v \in V$, there exists a $w \in V$ such that $v + w = 0$
8. $Iv = v$
9. $c(v + w) = cv + cw$ for all $c \in \mathbb{R}$
10. $(k + c)u = ku + cu$ for all $k, c \in \mathbb{R}$

The vector space that we most commonly work with is $\mathbb{R}^n$.

Practice

Show that the set $\mathbb{R}^n$ is a vector space.

Subspace

A subset U of V is called a subspace of V if it is also a vector space. To check if U is a subspace, you only need to check that the following properties hold:

1. **Additive Identity:** $\mathbf{0} \in U$
2. **Closed under addition:** $u + v \in U$ if $u, v \in U$
3. **Closed under multiplication:** if $a \in \mathbb{R}$ and $u \in U$, then $au \in U$

When we get to the proof sections, we will look at different subsets, and you will be asked to show whether different subsets are subspaces.

Example

Is the subset $\{\mathbf{0}\}$ where $\mathbf{0} \in \mathbb{R}^n$ a subspace of $\mathbb{R}^n$?
We need to check that the 3 properties above hold:

1. $\mathbf{0} \in \{\mathbf{0}\}$
2. $\mathbf{0} + \mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$
3. $a\mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$

   Since the properties hold, $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.