Assignment 1 Solutions

Due July 29, 2019

1. Consider the following system of linear equations:

$$\begin{aligned} -3x_1 - x_2 + 2x_3 &= 7 \\ 2x_2 - 2x_3 &= 8 \\ 6x_1 - 3x_2 + 6x_3 &= -9 \end{aligned}$$

   (a) Put the system of linear equations into an **augmented** matrix.

$$\left(\begin{array}{ccc|c} -3 & -1 & 2 & 7 \\ 0 & 2 & -2 & 8 \\ 6 & -3 & 6 & -9 \end{array}\right)$$

   (b) Find the reduced row echelon form of the **augmented** matrix.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

   (c) What is the rank of the **coefficient** matrix?

$$\text{rank} = 3$$

2. Consider the following system of linear equations:

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

   (a) Put the system of linear equations into a **coefficient** matrix.

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & 1 \\ 3 & 2 & 2 \\ -3 & 8 & 5 \end{pmatrix}$$

   (b) Find the reduced row echelon form of the **coefficient** matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) What is the dimension of the row space the **coefficient** matrix?

If A is the matrix in question, then $\dim(rowA) = 3$

3. What does the rank of a matrix tell us?

See the notes for what the rank of a matrix tell us.

Let A be a 3×3 matrix with $det(A) = 6$. Find each of the following if possible:

(a) $\det(A^T) = 6$

(b) We can't say what $\det(A+I)$ would be. Recall, that generally $\det(A+I) \neq \det(A) + \det(I)$.

(c)

$$\begin{aligned}
\det(3A) &= 3^3 a_{11}a_{22}a_{33} + 3^3 a_{12}a_{23}a_{31} + 3^3 a_{13}a_{21}a_{32} - 3^3 a_{31}a_{22}a_{13} - 3^3 a_{32}a_{23}a_{11} - 3^3 a_{33}a_{21}a_{12} \\
&= 3^3(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}) \\
&= 3^3|A| \\
&= 3^3 6 \\
&= 162
\end{aligned}$$

(d) $\det(A^4) = \det(A) \cdot \det(A) \cdot \det(A) \cdot \det(A) = 6^4 = 1296$

4. A property of traces is that $tr(AB) = tr(BA)$. Using this property, show that $tr(ABC) = tr(CAB) = tr(BCA)$.

Let $X = AB$, then $tr(XC) = tr(CX)$, which implies that $tr(ABC) = tr(CAB)$. Also, let $Y = CA$, then $tr(YB) = tr(BY)$, then $tr(CAB) = tr(BCA)$.

5. *This problem was taken from last year's problem set. It is such a great problem I felt that I needed to include it. Please do not look at last year's solution.*

Let X be a $n \times k$ real matrix. Define projection matrix $P := X(X'X)^{-1}X'$ and orthogonal matrix $M := I_n - P$. (You can assume $(X'X)^{-1}$ exists.)

(a) Show that P and M are symmetric and idempotent.

To show that P is idempotent, we need to show $PP = P$:

$$\begin{aligned}
PP &= X(X'X)^{-1}X'X(X'X)^{-1}X' \\
&= X(X'X)^{-1}X' \\
&= P
\end{aligned}$$

To show M is idempotent, we need to show $MM = M$:

$$\begin{aligned}
MM &= (I_n - P)(I_n - P) \\
&= I_nI_n - PI_n - I_nP + PP \\
&= I_n - P - P + P \\
&= I_n - P \\
&= M
\end{aligned}$$

To show that P is symmetric, we need to show that $P' = P$:

$$
\begin{aligned}
P' &= (X(X'X)^{-1}X')' \\
&= (X(X'X)^{-1}X')' \\
&= X'' \left((X'X)^{-1}\right)' X' \\
&= X \left((X'X)'\right)^{-1} X' \\
&= X \left(X'X''\right)^{-1} X' \\
&= X \left(X'X\right)^{-1} X' \\
&= P
\end{aligned}
$$

To show that M is symmetric, we need to show that $M' = M$:

$$
\begin{aligned}
M' &= (I_n - P)' \\
&= I_n' - P' \\
&= I_n - P \\
&= M
\end{aligned}
$$

(b) Show that $tr(P) = k$, $tr(M) = n - k$. Using problem 5's result, we see that $tr(X(X'X)^{-1}X') = tr(X'X(X'X)^{-1})$, and since $(X'X)(X'X)^{-1} = I_k$, it follows that $tr(P) = k$.

Since $M = I_n - P$, it follows that

$$
\begin{aligned}
tr(M) &= tr(I_n - P) \\
&= tr(I_n) - tr(P) \\
&= n - k
\end{aligned}
$$

6. Let V be defined as follows:

$$
V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}
$$

Surprise, surprise, V is not a vector space. Show by counterexample which properties (which are listed in the notes) are violated.

Property 2 is violated. Notice that if $\vec{v} \in V$ and $\vec{v} \neq \mathbf{0}$, then $c\vec{v} \notin V$ if $c < 0$.

Property 7 is violated. Notice that if $\vec{v} \in V$ and $\vec{v} \neq \mathbf{0}$, then there is not vector in V such that $\vec{v} + \vec{w} = \mathbf{0}$

7. Find the (real) eigenvalues and eigenvectors of the following matrix:

$$
\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}
$$

$$
\begin{aligned}
\lambda_1 &= -5, \lambda_2 = 9 \\
\vec{v}_1 &= \begin{bmatrix} -k \\ k \end{bmatrix} \\
\vec{v}_2 &= \begin{bmatrix} k \\ k \end{bmatrix}
\end{aligned}
$$

8. A diagonal matrix is a square matrix that has only zero value entries on the off-diagonal. Show that the eigenvalues of a diagonal matrix are the values on the diagonal of that matrix.

   To find the eigenvalues of a matrix, we need to find the determinant of $(A - \lambda I_n)$. The determinant of a diagonal matrix is just the product of the diagonal. Thus, the determinant is:

$$\det(A - \lambda I_n) = \prod_{i=1}^{n} (a_{ii} - \lambda_i)$$

   Notice that $\det(A - \lambda I_n) = \mathbf{0} \Rightarrow \lambda_i = a_{ii}$

9. The distance between two $n \times 1$ vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as:

$$dist(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

   Redefine this distance formula using the inner product.

$$dist(\mathbf{u}, \mathbf{v}) = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$$

10. List out all the elements of each set, and put the elements within curly brackets { and }.

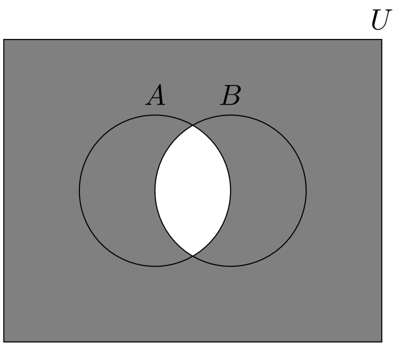
    (a) $A = \{n \in \mathbb{N} : 5 < n < 13\} = \{6, 7, 8, 9, 10, 11, 12\}$
    (b) $B = \{n \in \mathbb{Z} : |n^3| < 10\} = \{-2, -1, 0, 1, 2\}$
    (c) $C = \{x \in \mathbb{R} : x^2 + 1 = 0\} = \{\} = \emptyset$

11. Put the following sets in set builder notation. In other words, write each set in the form $\{f(x) \in \mathbb{Z} : p(x)\}$, where $f(x)$ is a function of x, and $p(x)$ is a condition of x.

    (a) $D = \{5, 6, 7, \ldots\} = \{n \in \mathbb{N} : n \geq 5\}$
    (b) $E = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\} = \{2^n : n \in \mathbb{Z}\}$
    (c) $F = \{-1, 0, 1, 16\} = \{(n - 2)^n : n \in \mathbb{N} \text{ and } n \leq 4\}$

12. Let U be a universal set, and let A and B be subsets of U. Draw a venn diagram for the following sets:

    (a) $\overline{A \cap B}$



(b) $\overline{A} \cap \overline{B}$