Assignment 2

Due August 5, 2019

Consider the $(\mathbb{R}, d_1)$ metric space.

1. Show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

2. Show that $A - (B \cap C) = (A - B) \cup (A - C)$.

3. State the negation of the following statements:

   (a) $\sqrt{3}$ is a rational number.

   (b) x is an even integer, or y is an odd integer.

4. Complete the following truth table:

P	Q	$\sim Q$	$P \wedge (\sim Q)$
T	T		
T	F		
F	T		
F	F		

5. Show that $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$ are logically equivalent.

6. Let $n \in \mathbb{Z}$. Prove that if n is even, then $7n - 9$ is odd.

7. Let $a, b, m \in \mathbb{Z}$. Prove that if $2a + 3b \geq 12m + 1$, then $a \geq 3m + 1$ or $b \geq 2m + 1$.

8. Let $a, b \in \mathbb{Z}$. Prove that if $a + b$ and ab are of the same parity (either both are even or both are odd), then a and b are even.

9. Disprove the following statement. If $x, y \in \mathbb{R}$, then $log(xy) = log(x) + log(y)$.

10. Let $X_1, X_2, \ldots, X_n$ be matrices where $n \in \mathbb{N}$. Using mathematical induction, show that $(X_1 X_2 \ldots X_n)^T = X_n^T \ldots X_2^T X_1^T$.

11. Using mathematical induction, show that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ where $n \in \mathbb{N}$.

12. A relation R is defined on $\mathbb{Z}$ by aRb if $|a - b| \leq 2$. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.

13. Verify that the following are valid metric spaces:
    (a) $(\mathbb{R}^2, \rho)$ where ρ is defined as:
    $$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$
    (b) $(\mathbb{R}^n, \ell^1)$
14. Show that the following are strictly increasing functions (like we did in class):
    (a) $f(x) = e^x + 2x$
    (b) $f(x) = x^3 - x^2$ where $x \geq 1$
15. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ (and a justification as to why) such that:
    (a) f is onto and one-to-one.
    (b) f is one-to-one but not onto.
    (c) f is onto but not one-to-one.

These last exercises (especially 17 and 18) are a bit harder than the ones you have seen before. You will probably spend the majority of your homework time on these exercises. If you don't understand what to do, please discuss with others approaches that you could take. I could care less if you write down a proof perfectly, I mostly care that you can express your thoughts intuitively and can work through a problem. So even if you can't solve the proof, put your thoughts down so I can evaluate your thought process.

Again, consider the $(\mathbb{R}, d_1)$ metric space.

16. Show (prove) that $\lim \frac{4n+5}{5n+2} = \frac{4}{5}$.
17. Prove that a Cauchy sequence in this metric space is bounded.
18. Prove the squeeze theorem. Or in other words, show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n$, then $\lim y_n = l$.