Assignment 3

Due August 12, 2019

For questions 1 and 2, consider the metric space $(\mathbb{R}, d_1)$ for the following problems, where d_1 is the absolute value metric.

1. Using the definition of open ball (or ε-neighborhood). Show that (a, b), where $a < b$, is an open set.

2. Let

$$B = \left\{ \frac{(-1)^n n}{n+1} : n \in \mathbb{N} \right\}$$

   (a) Find the limit points of B.

   (b) Is B a closed set? Why or why not?

   (c) Is B an open set? Why or why not?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Now assume that there is a $\lambda \in (0, 1)$ such that:

$$|f(x) - f(x')| \leq \lambda |x - x'|$$

   for all $x, x' \in \mathbb{R}$

   Suppose we start with $y_1 \in \mathbb{R}$ and construct a sequence (y_n) by a applying the function f at each index to the previous element of the sequence. Thus our sequence would look like the following:

$$\begin{aligned}(y_n) &= (y_1,\ y_2,\ \quad y_3,\ \qquad y_4,\ \qquad\quad ...) \\ &= (y_1, f(y_1), f(f(y_1)), f(f(f(y_1))), ...)\end{aligned}$$

   Or in other words, $y_{n+1} = f(y_n)$.
   You may find the following property of infinite series useful:

$$\sum_{i=1}^{\infty} ar^i = a \sum_{i=1}^{\infty} r^i = a \left(\frac{1}{1-r} \right)$$

   where $a \in \mathbb{R}$ and $r \in (0, 1)$. In other words, this infinite sum is less than the constant: $a \left(\frac{1}{1-r} \right)$.

   (a) Show that the sequence (y_n) is a Cauchy sequence.

   (b) Since (y_n) is a Cauchy sequence, we see that (y_n) is a convergent sequence, or in other words there is a limit point y such that $\lim_{n\to\infty} y_n = y$. Prove that y is a fixed point of f.

4. Determine the definiteness of the following symmetric matrix:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

For questions 5 and 6, let A be a convex subset of $\mathbb{R}^n$ where $f : A \to \mathbb{R}$. Let f be concave.

5. Show that f is quasiconcave.

6. Show that cf is a concave function when $c > 0$, and cf is a convex function when $c < 0$.

7. Consider the function $f(x) = \ln(1 + x)$.

   (a) Calculate $f(.5)$.

   (b) Using a first order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

   (c) Using a second order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

   (d) Using a third order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

8. Differentiate implicitly to find $\frac{dy}{dx}$:

$$x^2 - 3xy + y^2 - 2x + y - 5 = 0$$

9. Find the critical points and classify these as local max, local min, saddle point, or "can't tell":

$$f(x, y, z) = \left(x^2 + 2y^2 + 3z^2\right) e^{-\left(x^2+y^2+z^2\right)}$$

10. The Cobb-Douglas utility function is given by $U(x_1, x_2) = kx_1^{\alpha} x_2^{1-\alpha}$ where x_1 and x_2 are two goods. Assume a consumer's budget set is $p_1x_1 + p_2x_2 \leq I$. Do the following:

    (a) List the Karush Kuhn Tucker conditions for the problem above.

    (b) Solve for x_1, x_2, and the Lagrangian multiplier.

11. Consider the following production function: $y = f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$ for $i = 1, 2, \ldots, n$ where $a_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. The firm maximizes profits under perfect competition (in other words price or output, $p > 0$, and prices of inputs, $w_i > 0$, are exogenous or given):

$$\max_{x_1, x_2, \ldots, x_n} pf(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i$$

    (a) Solve for the maximizer $(x_1^*, x_2^*, \ldots, x_n^*)$

    (b) Show that x_i^* is homogeneous of degree 0 (for prices p and w's). What does this mean?