PhD Mathcamp Final **Name:** ______________________________
Summer 2019
Time Limit: 120 Minutes

Instructions: Some questions on this test may be a bit difficult. Relax, and answer all questions to the best of your ability (check every page to make sure you have answered everything). Note that partial solutions will receive partial credit, so putting something for a question will be better than leaving that question blank.

1. **(5 points)** Sets

   (a) Express the following set using set-builder notation $\{f(x) \in \mathbb{Z} : p(x)\}$, where $f(x)$ is a function of x, and $p(x)$ is a statement or condition of x.

$$\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...\}$$

   (b) Consider the metric space: $(\mathbb{R}, |\cdot|)$ where $|\cdot|$ is the absolute value metric. Is $(\frac{1}{2}, 1]$ open, closed, or neither? Justify your answer.

   Consider the metric space: $([0, 1], |\cdot|)$. Is $(\frac{1}{2}, 1]$ open, closed, or neither? Justify your answer.

2. **(15 points)** Consider the following consumer utility max problem (and assume price $m \geq p \geq 1$):

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \ln x_1 + x_2$$

such that

$$x_1 + px_2 \leq m$$
$$x_1 \geq 1$$
$$x_0 \geq 0$$

[1]

(a) Using the definition of strictly increasing function (without using derivatives), show that $f(x) = \ln x$ and $g(x) = x$ are strictly increasing functions.

(b) Calculate the Hessian, $D^2 f_{(x_1,x_2)}$, of the objective function, $f(x_1, x_2) = \ln x_1 + x_2$, and show that it is negative semi-definite over the domain $x_1 \geq 1, x_2 \geq 0$.

[1] Recall that $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$

(c) Define the Lagrangian and find the Karush-Kuhn-Tucker conditions (you don't have to include the nonnegativity constraint of $x_2 \geq 0$).

(d) In part (a) we showed that the objective function is strictly increasing in both inputs (x_1 and x_2). That means that the budget constraint ($x_1 + px_2 \leq m$) is binding, or in other words holds with equality ($x_1 + px_2 = m$). Give some intuition why this is the case.

(e) Using the conditions in (c) and (d), find the maximizers, x_1^* and x_2^*.

3. **(10 points)** Prove the following:

   (a) Recall that the epigraph of a function from $\mathbb{R} \to \mathbb{R}$ is the set of points lying on or above the graph: $epi\ f = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y \geq f(x)\}$. Show that the epigraph of a convex function is convex.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function. Using the definition of concave function, show that

$$\frac{f(y) - f(x)}{y - x} \leq f'(x)$$

Hint: You might find the definition of derivative helpful: $f'(x) = \frac{f(x+h)-f(x)}{h}$ *where* h *is some small value (we take* $h \to 0$ *to find the derivative)*

4. **(5 points)** Show that a convergent sequence in the metric space $(\mathbb{R}, |\cdot|)$ where $|\cdot|$ is the absolute value metric is a Cauchy sequence.

5. **(5 points)** Consider the metric space $(\mathbb{R}, |\cdot|)$ where $|\cdot|$ is the absolute value metric. The Brouwer's Fixed Point Theorem is as follows:

   Suppose that $X \subset \mathbb{R}$ is a nonempty, compact, convex set, and that $f : X \to X$ is a continuous function from X into itself. Then f has a fixed point; that is an $x \in X$ such that $x = f(x)$.

   Use the intermediate value theorem to prove Brouwer's Fixed Point Theorem in this metric space.