Mathematics Bootcamp Assignment 0

May 25, 2018

There are 43 problems in this assignment. Take a deep breath. That is a lot, but many are quite quick if you have taken courses in calculus. The questions are not meant to be exhaustive of all important topics in univariate calculus. Instead, they are intended to *nudge* you to remember things you likely once learned, but forgot. For example, the rules of limits, l'Hopital's rule, continuity, the product rule, the chain rule, the fundamental theorem of calculus, taylor series, integration, the intermediate value theorem, the mean value theorem, inverse functions, implicit differentiation, etc.

1 Calculus

Problem 1 Compute

$$\lim_{x \to 3} \frac{5x^2 - 8x - 13}{x^2 - 5}$$

Problem 2 Compute

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$

Problem 3 Compute

$$\lim_{x \to 4} \frac{3 - \sqrt{x + 5}}{x - 4}$$

Problem 4 Consider the values of constants a and b so that $\lim_{x \to 2} f(x)$ exists and is equal to $f(2)$ where $f(x)$ is defined as below.

$$f(x) = \begin{cases} a + bx & \text{if } x > 2 \\ 3 & \text{if } x = 2 \\ b - ax^2 & \text{if } x < 2 \end{cases}$$

Problem 5 Compute the following limit.

$$\lim_{x \to \infty} \frac{100}{x^2 + 5}$$

Problem 6 Compute

$$\lim_{x \to \infty} (3x^3 - 1000x^2)$$

Problem 7 Compute

$$\lim_{x\to\infty} \frac{7x^2 + x - 100}{2x^2 - 5x}$$

Problem 8 Compute

$$\lim_{x\to\infty} \left(3^x + 3^{2x}\right)^{\frac{1}{x}}$$

Problem 9 Compute $\lim_{x\to 0^+} x \cdot \ln x$.

Problem 10 Compute $\lim_{x\to 0^+} x \cdot (\ln x)^2$.

Problem 11 Compute $\lim_{x\to 0}(1 - x)^{1/x}$.

Problem 12 Consider the following functions for $w \geq 0$ and $0 < \sigma < 1$.

$$u(w) = \frac{w^{1-\sigma} - 1}{1 - \sigma}$$
$$u(w) = \ln(w)$$
$$u(w) = \sqrt{w}$$

For each function above, find the following limits (if they exist) where $u'(\cdot)$ and $u''(\cdot)$ represent the first and second derivatives respectively.

(i) $\lim_{w\to 0} -u''(w)/u'(w)$.

(ii) $\lim_{w\to\infty} -u''(w)/u'(w)$.

(iii) $\lim_{w\to 0} (-u''(w) \cdot w)/u'(w)$.

(iv) $\lim_{w\to\infty} (-u''(w) \cdot w)/u'(w)$.

Problem 13 Differentiate $y = x^x$ (with respect to x).

Problem 14 Differentiate $y = x^{e^x}$ (with respect to x).

Problem 15 Compute the following limits associated with the functions $f(x) = |x|$.

$$\lim_{h\to 0} \frac{f(-2 + h) - f(-2)}{h}$$
$$\lim_{h\to 0} \frac{f(0 + h) - f(0)}{h}$$
$$\lim_{h\to 0} \frac{f(3 + h) + f(3)}{h}$$

Problem 16 Define the intervals (if any) over which the following function is continuous.

$$f(x) = \frac{7x^5 + x - 2}{x^2 - 4}$$

Problem 17   Show that there is a root of the equation $3x^7 - 2x^5 + x - 1 = 0$ between 0 and 1.

Problem 18   Compute the following limit associated with $f(x) = x^2 - 8x + 9$.

$$\lim_{x \to a} \frac{(x^2 - 8x + 9) - (a^2 - 8a + 9)}{x - a}$$

Problem 19   If the function $f(x)$ is differentiable in the interval (a, b) and $|f'(x)| \le B < \infty$ for all x in the interval (a, b) then is the maximum change in the function over any sub interval $(c, d) \subseteq (a, b)$ finite or infinite? Prove it.

Problem 20   If the function $f(x)$ is differentiable on (a, b), but not continuously differentiable, then is f continuous everywhere on (a, b)? Prove it. (Hint: use a theorem)

Problem 21   Can you differentiate the expression $2x = 1$? If so what is the derivative? If not, why not?

Problem 22   Find the derivative of the function

$$f(x) = \frac{\sqrt[4]{x}}{x^{-1} \sqrt{x^5}}$$

Problem 23   Differentiate the function

$$f(x) = \frac{3x^2 - 5\sqrt{x}}{6x^4}$$

Problem 24   Find the linearization of the function $f(x) = \sqrt[3]{1 + x}$ at $a = 0$ and use it to approximate the numbers $f(-0.05)$ and $f(0.1)$. Are these approximations overestimates or underestimates?

Problem 25   Let $f(x) = (3x - 5)/(4 - 2x)$ and find $f^{-1}(x)$. Then compare $f'(x)$ and $\frac{d}{dx} f^{-1}(x)$ and describe the relationship (if any).

Problem 26   Compute the derivative of the following function, $H(p)$, with respect to p and use it to show the answer to the following questions.

$$H(p) = p \log_2 \left(\frac{1}{p} \right) + (1 - p) \log_2 \left(\frac{1}{1 - p} \right)$$

(i) Is there a global maximum and minimum over the interval $[0, 1]$, if so, what is it?

(ii) If so, over what subset of $[0, 1]$, if any, is the function increasing?

(iii) If so, over what subset of $[0, 1]$, if any, is the function decreasing?

Problem 27 Consider the function $f(x) = x^4 e^x$ with domain all real numbers.

(i) Find the x-value(s) of all roots (x-intercepts) of f.

(ii) Find the x- and y-value(s) of all critical points and identify each as a local max, local min, or neither.

(iii) Find the x- and y-value(s) of all global extrema and identify each as a global max or global min.

(iv) Find the x-value(s) of all inflection points.

Problem 28 Use the Intermediate Value Theorem to show that $f(x) = x^3 - 2x - 1$ has a root on $[1, 2]$.

Problem 29 Does the Extreme Value Theorem say anything about the function $f(x) = x^2$ on each of the following intervals? If so what does it say? In either case, explain why.

(i) $[1, 4]$

(ii) $(1, 4)$

Problem 30 Find the value of the constant c that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on the interval $[0, 3]$.

Problem 31 For the equation $x^3 + y^3 = \ln(xy) - 1$ use implicit differentiation to find dy/dx.

Problem 32 Consider the function

$$f(x) = \begin{cases} b - x^2 & \text{if } x < 3 \\ ax & \text{if } x \geq 3 \end{cases}$$

(i) What condition(s) must be placed on the constants a and b in order for f to be continuous on $(-\infty, \infty)$?

(ii) For what values of the constants a and b will f be differentiable on $(-\infty, \infty)$?

Problem 33 Consider the function $f(x) = \ln(x^2)$. Find the fourth-order Taylor polynomial for $f(x)$ centered at $x_0 = 1$.

Problem 34 Let X be a random variable with a probability density function (PDF) $f(x)$

$$f(x) = \begin{cases} ce^{-x/3} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Remember that a PDF has the property that $\int_{\infty}^{\infty} f(x) \, dx = 1$.

(i) Find the value of the constant c that makes $f(x)$ a valid PDF.

(ii) Find the probability that $X \leq 1/4$.

Problem 35 Find the derivative of the following function

$$G(x) = \int_1^{\sin x} t \, dt$$

Problem 36 Find the derivative of the following function

$$G(x) = \int_{72}^{\ln(x)} t \, dt$$

Problem 37 Suppose that $\sum_{n=1}^{\infty} a_n$ represents a convergent series and that no term of the series equals zero, i.e., $a_n \neq 0$ for all $n = 1, 2, \ldots$. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.

Problem 38 Determine whether the given sequence is increasing, decreasing, or not monotonic. Is the sequence bounded? On the basis of what you find, does the series converge, diverge or can't be determined?

$$a_n = \frac{1}{5^n}$$

Problem 39 Differentiate the function $y = 3(x^2 - 1)^3(x^2 + 1)^5$.

Problem 40 Consider a continuously differentiable utility function $u(\cdot)$ such that $u' > 0$ and $u'' < 0$. Utility comes from income I which takes on two different values $I_H(x)$ and $I_L(x)$ and that the probability of that $I = I_H$ is given by $p(x)$ where $x \geq 0$ and $p'(x) > 0$. Differentiate the expected utility with respect to x.

$$E[u] = p(x)u[I_H(x)] + (1 - p(x))u[I_L(x)]$$

Problem 41 Consider a twice continuously differentiable utility function $U(\cdot)$ such that $U' > 0$ and $U'' < 0$. Utility comes from wealth W and a person takes a gamble h that represents a gain or loss to the a person's wealth and $h = 0$ on average ($E(h) = 0$). Let p represent the size of an insurance premium paid to avoid taking the gamble and makes the person exactly indifferent between the gamble h and paying p with certainty. Let E represent the expectation operator. Use the identity below to answer the questions.

$$E[U(W + h)] \equiv U(W - p)$$

(i) Derive a linear (first order) Taylor approximation of the right-hand side of the identity (with respect to W).

(ii) Derive a quadratic (second order) Taylor approximation of the left-hand side of the identity (with respect to W).

(iii) Let $r(W) = -U''(W)/U'(W)$ (this is the Arrow-Pratt measure of risk-aversion) use the previous approximations and the identity relationship to sign the derivative $dP/dr(W)$. What is the sign and what relationship does this imply about the level of risk aversion and the amount of insurance premium an individual is willing to pay?

Problem 42 A new machine has a rental rate $v(s)$ at any time s and depreciates at rate d. With interest rate r the present discounted value of this machine is

$$PDV(t) = \int_t^\infty e^{(r+d)t} v(s) e^{-(r+d)s} \, ds$$

Let $p(t)$ be the purchase price of the machine at time t. In equilibrium the purchase price at time t, $p(t)$ will equal the present discounted value and hence we have

$$p(t) = \int_t^\infty e^{(r+d)t} v(s) e^{-(r+d)s} \, ds$$

(i) Using the last expression for $p(t)$, find $\frac{dp(t)}{dt}$.

(ii) Using your expression for $\frac{dp(t)}{dt}$ solve for $v(t)$ in terms of $p(t)$ to show the relationship between a new machines rental rate at time t and the machine's purchase price at time t.

Problem 43 A forester must decide when to cut down a growing tree. The value at any time t is given by $f(t)$ where $f' > 0$ and $f'' < 0$ and there was an initial investment of L. The continuous interest rate is r. The forester must choose t (the time of harvest) to maximize the present discounted value of her profits.

$$PDV(t) = e^{-rt} f(t) - L$$

(i) Differentiate $PDV(t)$ with respect to t.

(ii) Use the derivative as a first order condition to characterize the relationship between the t that maximizes the forester's present discounted value of profit and the interest rate r.