# Assignment 1

July 25, 2018

1. Solve the following system of linear equations:

$$\begin{aligned} x - 3z &= -2 \\ -3x + y + 6z &= 3 \\ 2x - 2y - z &= -1 \end{aligned}$$

2. Solve the following system of linear equations:

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

3. Suppose $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$. Find $A$.

4. Show that $A^{-1} = (A^T A)^{-1} A^T$

5. Evaluate the following determinants

$$\text{(a)} \begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$$

6. Invert the following matrices, then show that the inverted matrices are actually inverted (i.e. $A'A = I$).

$$\text{(a)} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$\text{(c)} \begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

7. Let $A$ be a $3 \times 3$ matrix with $\det(A) = 5$. Find each of the following if possible.

   (a) $\det(A^T)$

   (b) $\det(A + I)$

   (c) $\det(2A)$.

8. Show that if $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

9. Let $A$ be an $n \times n$ invertible matrix; and $D$ and $CA^{-1}B$ be square, $n \times n$ matrices. Show that:

$$det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(A)det(D - CA^{-1}B)$$

Hint: Use the following matrix decomposition:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

10. Let $A$ and $B$ be square matrices. Show that $tr(AB) = tr(BA)$.

11. Let $X$ be a $n \times k$ real matrix. Define projection matrix $P := X(X'X)^{-1}X'$ and orthogonal matrix $M := I_n - P$. (You can assume $(X'X)^{-1}$ exists.)

    (a) Show that $P$ and $M$ are symmetric and idempotent.

    (b) Show that $tr(P) = k$, $tr(M) = n - k$.

12. Consider the following equation: $(y - Xb)'(y - Xb)$ where $y$ is an $n \times 1$ vector, $X$ is an $n \times k$ matrix, and $b$ is a $k \times 1$ vector.

    (a) What is the size of $(y - Xb)'(y - Xb)$?

    (b) Simplify the expression above using distributive properties of matrix algebra.

    (c) Take the derivative of what you found above with respect to $b$.

    (d) Now solve for $b$.

    (e) What is the size of the answer you found in d?