Assignment 1 Solutions

July 31, 2018

1. We can write the system of equations into an augmented matrix.

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ -3 & 1 & 6 & | & 3 \\ 2 & -2 & -1 & | & -1 \end{bmatrix}$$

Putting the matrix into reduced row echelon form yields the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & | & 7 \\ 1 & 1 & 0 & | & 6 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

Thus the solution to the system of linear equations is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}$$

2. We can write the system of equations into an augmented matrix.

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ -2 & 2 & 1 & | & 4 \\ 3 & 2 & 2 & | & 5 \\ -3 & 8 & 5 & | & 17 \end{bmatrix}$$

Putting the matrix into reduced row echelon form yields the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & \frac{3}{2} \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus the solution to the system of linear equations is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

3. Doing a little matrix algebra, we can solve for A:

$$\begin{aligned} AB &= AB \\ ABB^{-1} &= ABB^{-1} \\ A &= ABB^{-1} \end{aligned} \tag{1}$$

Thus, to find A, we need to first calculate B^{-1} and then post multiply it to AB.

$$\begin{aligned} B^{-1} &= \frac{1}{7-6}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \end{aligned}$$

Now we can solve for A using equation (1):

$$\begin{aligned} A &= ABB^{-1} \\ &= \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix} \end{aligned}$$

4. We can simplify the RHS:

$$\begin{aligned} (A'A)^{-1}A' &= A^{-1}A'^{-1}A' \\ &= A^{-1}I \\ &= A^{-1} \end{aligned}$$

If you want to start from the LHS:

$$\begin{aligned} A^{-1} &= A^{-1}A'^{-1}A' \\ &= (A'A)^{-1}A' \end{aligned}$$

5. (a) We can use the shortcut (found in Simon and Blume) to calculate the determinant:

$$\begin{aligned} \begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix} &= 4\cdot 1\cdot(-4) + 3\cdot 2\cdot 5 + 0 - 0 - 4\cdot 2\cdot(-1) - 3\cdot 3\cdot(-4) \\ &= 58 \end{aligned}$$

(b) We will have to use the method described in the notes. I can use row 2 to simplify the calculation for the determinant:

$$\begin{aligned} \begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix} &= 1^{2+2}\cdot\begin{vmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} \\ &= 12 - 2 - 2 \\ &= 8 \end{aligned}$$

6. (a) $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 0 & -5 \\ 0 & \frac{1}{3} & 0 \\ -1 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ -2 & -1 & -1 \\ \frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}$

7. We can use the properties defined in the notes to help us calculate the following determinants:

(a) $\det(A^T) = \det(A) = 5$

(b) We cannot say anything about $\det(A+I)$ since we do not know what A looks like. Recall, that generally $\det(A+I) \neq \det(A) + det(I)$

(c) If we scalar multiply matrix A by 2, then :

$$\begin{aligned}
|2A| &= 2^3 a_{11}a_{22}a_{33} + 2^3 a_{12}a_{23}a_{31} + 2^3 a_{13}a_{21}a_{32} - 2^3 a_{31}a_{22}a_{13} - 2^3 a_{32}a_{23}a_{11} - 2^3 a_{33}a_{21}a_{12} \\
&= 2^3(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}) \\
&= 2^3|A| \\
&= 2^3 5 \\
&= 40
\end{aligned}$$

8. Notice that $AA^{-1} = I$, thus:

$$\begin{aligned}
\det(AA^{-1}) &= \det(1) \\
\det(A)\det(A^{-1}) &= 1 \\
\det(A^{-1}) &= \frac{1}{\det(A)}
\end{aligned}$$

9. Using the matrix decomposition given in the hint, we see that:

$$\begin{aligned}
\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}\right) \\
&= \det\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \det\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \\
&= \det(I)\det\left(A\left(D - CA^{-1}B\right)\right) \\
&= \det(I)\det(A)\det\left(D - CA^{-1}B\right) \\
&= \det(A)\det\left(D - CA^{-1}B\right)
\end{aligned}$$

10. If both A and B are $n \times n$ matrices, then we see that:

$$\begin{aligned}
tr(AB) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} \\
tr(BA) &= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}a_{ij}
\end{aligned}$$

If you calculate $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}$ and $\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}a_{ij}$, you will find that $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}a_{ij}$, thus, $tr(AB) = tr(BA)$.

11. (a) To show that P is idempotent, we need to show $PP = P$:

$$\begin{aligned}
PP &= X(X'X)^{-1}X'X(X'X)^{-1}X' \\
&= X(X'X)^{-1}X' \\
&= P
\end{aligned}$$

To show M is idempotent, we need to show $MM = M$:

$$\begin{aligned} MM &= (I_n - P)(I_n - P) \\ &= I_n I_n - P I_n - I_n P + PP \\ &= I_n - P - P + P \\ &= I_n - P \\ &= M \end{aligned}$$

To show that P is symmetric, we need to show that $P' = P$:

$$\begin{aligned} P' &= (X(X'X)^{-1}X')' \\ &= (X(X'X)^{-1}X')' \\ &= X'' \left((X'X)^{-1}\right)' X' \\ &= X \left((X'X)'\right)^{-1} X' \\ &= X \left(X'X''\right)^{-1} X' \\ &= X \left(X'X\right)^{-1} X' \\ &= P \end{aligned}$$

To show that M is symmetric, we need to show that $M' = M$:

$$\begin{aligned} M' &= (I_n - P)' \\ &= I_n' - P' \\ &= I_n - P \\ &= M \end{aligned}$$

(b) First, I need to present a lemma. If A is $m \times n$, B is $n \times k$, and C is $k \times m$, then $tr(ABC) = tr(CAB)$:

$$\begin{aligned} tr(AB) &= \sum_{j=1}^{k} (AB)_{jjj} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{k} (A_{ij} B_{ji}) \\ &= \sum_{j=1}^{k} \sum_{i=1}^{n} (A_{ji} B_{ij}) \\ &= tr(BA) \end{aligned}$$

By induction, if we define $D = (AB)$, then it follows that $tr(DC) = tr(CD)$, which in turn means that $tr(ABC) = tr(CAB)$.
Noting this lemma, we see that $tr(X(X'X)^{-1}X') = tr(X'X(X'X)^{-1})$, and since $(X'X)(X'X)^{-1} = I_k$, it follows that $tr(P) = k$.

Since $M = I_n - P$, it follows that

$$\begin{aligned} tr(M) &= tr(I_n - P) \\ &= tr(I_n) - tr(P) \\ &= n - k \end{aligned}$$

12. (a) Notice that the size of $(y - Xb)$ is $n \times 1$, and the size of $(y - Xb)'$ is $1 \times n$. Thus The size of $(y - Xb)'(y - Xb)$ is 1×1 (or in other words is a scalar).

(b) Simplifying the equation yields:

$$\begin{aligned}(y - Xb)'(y - Xb) &= y'y - b'Xy - y'Xb + b'X'Xb \\ &= y'y - 2b'Xy + b'X'Xb\end{aligned} \tag{2}$$

Notice that in equation (2), $b'Xy$ and $y'Xb$ are both scalars, thus $b'Xy = y'Xb$.

(c) Taking the derivative of (b) with respect to b yields:

$$\frac{\partial \left(y'y - 2b'Xy + b'X'Xb\right)}{\partial b} = -2X'y + 2X'Xb$$

(d) An important detail that was not included was that we are minimizing the equation $(y - Xb)'(y - Xb)$ with respect to b. Thus, from (c) we can use this information to equate $\frac{\partial\left(y'y-2b'Xy+b'X'Xb\right)}{\partial b} = 0$. Solving for b yields:

$$\begin{aligned}-2X'y + 2X'Xb &= 0 \\ (X'X)^{-1}(X'X)b &= (X'X)^{-1}X'y \\ b &= (X'X)^{-1}X'y\end{aligned}$$

(e) Notice that since $(X'X)^{-1}$ is $k \times k$, X' is $k \times n$, and y is $n \times 1$, we see that b is $k \times 1$.