PhD Mathcamp Final **Name:** ______________________
Summer 2018
Time Limit: 120 Minutes

Instructions: Some questions on this test may be a bit difficult. Relax, and answer all 5 questions to the best of your ability (check every page to make sure you have answered everything). Note that partial solutions will receive partial credit, so putting something for a question will be better than leaving that question blank.

1. **(10 points)** Let $U = x^\alpha y^\beta$ where $(x, y) \in \mathbb{R}^2_+$, $\alpha + \beta = 1$, and $\alpha, \beta > 0$.[1]

   (a) Is U homogeneous? If so, of what degree?

$$\begin{aligned} U(tx, ty) &= (tx)^\alpha (ty)^\beta \\ &= t^\alpha t^\beta x^\alpha y^\beta \\ &= t^{\alpha+\beta} x^\alpha y^\beta \\ &= t x^\alpha y^\beta \\ &= tU(x, y) \end{aligned}$$

   So U is homogeneous of degree 1.

   (b) Show that U is concave.

$$DU = \begin{bmatrix} \alpha x^{\alpha-1} y^\beta & \beta x^\alpha y^{\beta-1} \end{bmatrix}$$
$$D^2U = \begin{bmatrix} \alpha(\alpha-1)x^{\alpha-2}y^\beta & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^\alpha y^{\beta-2} \end{bmatrix}$$

   Notice that the first order leading principal minor is ≥ 0 since:

$$\alpha(\alpha-1)x^{\alpha-2}y^\beta \geq 0$$

   So we need to look at all of the principal minors.

   <u>First Order Principal Minors:</u>
   $\alpha(\alpha-1)x^{\alpha-2}y^\beta \leq 0$ $\qquad$ $\beta(\beta-1)x^\alpha y^{\beta-2} \leq 0$

   <u>Second Order Principal Minor:</u>
   $|D^2U| = \alpha(\alpha-1)\beta(\beta-1)x^{2\alpha-2}y^{2\beta-2} - \alpha^2\beta^2 x^{2\alpha-2}y^{2\beta-2}$
   $|D^2U| = \alpha\beta(1-\alpha-\beta)x^{2\alpha-2}y^{2\beta-2} = 0$ $\qquad$ since $\alpha + \beta = 1$.

   D^2U is negative semidefinite. Thus U is concave.

[1] Recall that $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$

2. **(5 points)** Express the following sets using set-builder notation $\{f(x) \in \mathbb{Z} : p(x)\}$, where $f(x)$ is a function of x, and $p(x)$ is a statement or condition of x.

   (a) {2,4,6,8,...}

   $$\{2n \in \mathbb{Z} : n > 0\}$$

   (b) {-27,-8,-1,0}

   $$\{n^3 \in \mathbb{Z} : -3 \leq n \leq 0\}$$

3. **(5 points)** Consider the metric space: $(\mathbb{R}, |\cdot|)$.

   (a) Show that the interval $(0, 1)$ is open.

   Notice that the complement of $(0, 1)$: $\overline{(0,1)} = (-\infty, 0] \cup [1, \infty)$. $(-\infty, 0] \cup [1, \infty)$ is closed since it contains all of its limit points (since $(\mathbb{R}, |\cdot|)$ is the metric space we are working in). Thus $(0, 1)$ is open.

   <u>Alternative Solution:</u> See assignment 3 solutions.

   (b) Show that the set $\{1, 2, 3\}$ is closed.

   Each point in the set $\{1, 2, 3\}$ is an isolated points. Notice that isolated points are not limit points, thus the set of limit points is: $\{\} = \emptyset$. Since $\emptyset \subseteq \{1, 2, 3\}$, $\{1, 2, 3\}$ is closed.

4. **(10 points)** Prove the following:

   (a) The intersection of two convex sets is convex.

   Let A and B be two convex sets.
   Now let $x, y \in A \cap B$.
   Thus $x, y \in A$ and $x, y \in B$.
   Since $x, y \in A$, it follows that $\lambda x + (1-\lambda)y \in A$ for $\lambda \in [0,1]$ as A is convex.
   Since $x, y \in B$, it follows that $\lambda x + (1-\lambda)y \in B$ for $\lambda \in [0,1]$ as B is convex
   Finally, $\lambda x + (1-\lambda)y \in A$ and $\lambda x + (1-\lambda)y \in B$ implies that $\lambda x + (1-\lambda)y \in A \cap B$.
   Therefore $A \cap B$ is convex.

(b) The maximum of two convex functions is convex. In other words, if the functions f_1 and f_2 are convex, then $\max\{f_1, f_2\}$ is convex.

This question will not be graded.

Let f_1 and f_2 be convex functions and let $f(x) = \max\{f_1(x), f_2(x)\}$. Thus:

$$f_i(\alpha x + (1-\alpha)y) \leq \alpha f_i(x) + (1-\alpha)f_i \tag{1}$$

for any $\alpha \in [0,1]$ and any $x, y \in \text{Dom}(f_i)$ where $i \in \{1,2\}$
Taking the max of both sides of (1), we get the following:

$$\begin{aligned}
&\max_i\{f_i(\alpha x + (1-\alpha)y)\} \leq \max_i\{\alpha f_i(x) + (1-\alpha)f_i(y)\} \\
\Rightarrow &\max_i\{f_i(\alpha x + (1-\alpha)y)\} \leq \max_i\{\alpha f_i(x)\} + \max_i\{(1-\alpha)f_i(y)\} \\
\Rightarrow &\max_i\{f_i(\alpha x + (1-\alpha)y)\} \leq \alpha \max_i\{f_i(x)\} + (1-\alpha)\max_i\{f_i(y)\} \\
\Rightarrow &f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)
\end{aligned}$$

Therefore, f is convex.

Alternative Solution: Notice that we could use the theorem that says that the epigraph of a function is a convex set iff the function is convex. When we take the maximum of two functions, we are taking an intersection of the two epigraphs (of f_1 and f_2) to form the epigraph of the newly created function ($\max\{f_1, f_2\}$). Since the two functions, f_1 and f_2, are convex, their epigraphs are convex. The intersection of the epigraphs of f_1 and f_2 is convex (as shown in 4a). This intersection is the epigraph of $\max\{f_1, f_2\}$. Thus the function defined by $\max\{f_1, f_2\}$ is convex.

5. **(10 points)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Now assume that there is a $\lambda \in (0, 1)$ such that:

$$|f(x) - f(x')| \leq \lambda |x - x'|$$

for all $x, x' \in \mathbb{R}$

Suppose we start with $y_1 \in \mathbb{R}$ and construct a sequence (y_n) by a applying the function f at each index to the previous element of the sequence. Thus our sequence would look like the following:

$$\begin{aligned}(y_n) &= (y_1,\ y_2,\quad\ y_3,\qquad\ y_4,\qquad\qquad ...)\\ &= (y_1, f(y_1), f(f(y_1)), f(f(f(y_1))), ...)\end{aligned}$$

Or in other words, $y_{n+1} = f(y_n)$.
You may find the following property of infinite series useful:

$$\sum_{i=1}^{\infty} ar^i = a \sum_{i=1}^{\infty} r^i = a\left(\frac{1}{1-r}\right)$$

where $a \in \mathbb{R}$ and $r \in (0, 1)$. In other words, this infinite sum is less than the constant: $a\left(\frac{1}{1-r}\right)$.

(a) Show that the sequence (y_n) is a Cauchy sequence.

Notice that you are acutally proving the contraction mapping theorem in $(\mathbb{R}, |\cdot|)$, yay!
We need to show that for $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, n \geq N$, it follows that

$$|y_m - y_n| < \varepsilon$$

Assume without loss of generality that $n > m$ where $m, n \in \mathbb{N}$:

$$\begin{aligned}|y_{m+1} - y_{m+2}| &= |f(y_m) - f(y_{m+1})|\\ &\leq \lambda\,|y_m - y_{m+1}|\end{aligned}$$

where $\lambda \in (0, 1)$ Therefore:

$$\begin{aligned}|y_{m+1} - y_{m+2}| &\leq \lambda\,|y_m - y_{m+1}|\\ &\leq \lambda^2\,|y_{m-1} - y_m|\\ &\leq \lambda^3\,|y_{m-2} - y_{m-1}|\\ &\ \vdots\\ &\leq \lambda^m\,|y_1 - y_2|\end{aligned}$$

Thus $|y_{m+1} - y_{m+2}| \leq \lambda^m\,|y_1 - y_2|$
Therefore:

$$\begin{aligned}|y_m - y_n| &\leq |y_m - y_{m+1} + y_{m+1} - y_{m+2} + y_{m+2} - ... + y_{n-1} - y_n|\\ &\leq |y_m - y_{m+1}| + |y_{m+1} - y_{m+2}| + ... + |y_{n-1} - y_n|\\ &\leq \lambda^{m-1}\,|y_1 - y_2| + \lambda^m\,|y_1 - y_2| + \ldots + \lambda^{n-2}\,|y_1 - y_2|\\ &= \lambda^{m-1}(1 + \lambda + \lambda^2 + \ldots + \lambda^{n-m-1})\,|y_1 - y_2|\\ &< \lambda^{m-1}\left(\frac{1}{1-\lambda}\right)|y_1 - y_2|\end{aligned}$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that:

$$\lambda^{N-1} < \frac{\varepsilon(1-\lambda)}{|y_1 - y_2|}$$

Then for $n > m \geq N$, we see that:

$$|y_1 - y_2| < \varepsilon$$

Thus (y_n) is cauchy.

(b) Since (y_n) is a Cauchy sequence, we see that (y_n) is a convergent sequence, or in other words there is a limit point y such that $\lim_{n\to\infty} y_n = y$. Prove that y is a fixed point of f.

Notice that $\lim_{n\to\infty} y_n = y$ and also $\lim_{n\to\infty} y_{n+1} = y$.
Since $y_{n+1} = f(y_n)$, it follows that $\lim_{n\to\infty} f(y_n) = y$.
Thus $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_n = y$.
In other words, $f(y) = y$, so y is a fixed point.