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Sets

In this initial chapter, you will be introduced to, or more than likely be reminded of, a fundamental idea that occurs throughout mathematics: sets. Indeed, a *set* is an object from which every mathematical structure is constructed (as we will often see in the succeeding chapters). Although there is a formal subject called *set theory* in which the properties of sets follow from a number of axioms, this is neither our interest nor our need. It is our desire to keep the discussion of sets informal without sacrificing clarity. It is almost a certainty that portions of this chapter will be familiar to you. Nevertheless, it is important that we understand what is meant by a set, how mathematicians describe sets, the notation used with sets, and several concepts that involve sets.

You've been experiencing sets all your life. In fact, all of the following are examples of sets: the students in a particular class who have an iPod, the items on a shopping list, the integers. As a small child, you learned to say the alphabet. When you did this, you were actually listing the letters that make up the set we call the alphabet. A **set** is a collection of objects. The objects that make up a set are called its **elements** (or **members**). The elements of a softball team are the players; while the elements of the alphabet are letters.

It is customary to use capital (upper case) letters (such as A, B, C, S, X, Y) to designate sets and lower case letters (for example, a, b, c, s, x, y) to represent elements of sets. If a is an element of the set A, then we write $a \in A$; if a does not belong to A, then we write $a \notin A$.

1.1 Describing a Set

There will be many occasions when we (or you) will need to describe a set. The most important requirement when describing a set is that the description makes it clear precisely which elements belong to the set.

If a set consists of a small number of elements, then this set can be described by explicitly listing its elements between braces (curly brackets) where the elements are separated by commas. Thus $S = \{1, 2, 3\}$ is a set, consisting of the numbers 1, 2 and 3. The order in which the elements are listed doesn't matter. Thus the set S just mentioned could be written as $S = \{3, 2, 1\}$ or $S = \{2, 1, 3\}$, for example. They describe the same set. If a set T consists of the first five letters of the alphabet, then it is not essential that we write $T = \{a, b, c, d, e\}$; that is, the elements of T need not be listed in alphabetical order. On the other hand, listing the elements of T in any other order may create unnecessary confusion.

The set A of all people who signed the Declaration of Independence and later became president of the United States is $A = \{$John Adams, Thomas Jefferson$\}$ and the set B of all positive even integers less than 20 is $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$. Some sets contain too many elements to be listed this way. Perhaps even the set B just given contains too many elements to describe in this manner. In such cases, the ellipsis or "three dot notation" is often helpful. For example, $X = \{1, 3, 5, \ldots, 49\}$ is the set of all positive odd integers less than 50, while $Y = \{2, 4, 6, \ldots\}$ is the set of all positive even integers. The three dots mean "and so on" for Y and "and so on up to" for X.

A set need not contain any elements. Although it may seem peculiar to consider sets without elements, these kinds of sets occur surprisingly often and in a variety of settings. For example, if S is the set of real number solutions of the equation $x^2 + 1 = 0$, then S contains no elements. There is only one set that contains no elements, and it is called the **empty set** (or sometimes the **null set** or **void set**). The empty set is denoted by $\emptyset$. We also write $\emptyset = \{\ \}$. In addition to the example given above, the set of all real numbers x such that $x^2 < 0$ is also empty.

The elements of a set may in fact be sets themselves. The symbol ♦ below indicates the conclusion of an example.

Example 1.1 *The set* $S = \{1, 2, \{1, 2\}, \emptyset\}$ *consists of four elements, two of which are sets, namely,* $\{1, 2\}$ *and* $\emptyset$. *If we write* $C = \{1, 2\}$, *then we can also write* $S = \{1, 2, C, \emptyset\}$.

The set $T = \{0, \{1, 2, 3\}, 4, 5\}$ *also has four elements, namely, the three integers* $0, 4$ *and* 5 *and the set* $\{1, 2, 3\}$. *Even though* $2 \in \{1, 2, 3\}$, *the number* 2 *is not an element of* T; *that is,* $2 \notin T$. ♦

Often sets consist of those elements satisfying some condition or possessing some specified property. In this case, we can define such a set as $S = \{x : p(x)\}$, where, by this, we mean that S consists of all those elements x satisfying some condition $p(x)$ concerning x. Some mathematicians write $S = \{x \mid p(x)\}$; that is, some prefer to write a vertical line rather than a colon (which, by itself here, is understood to mean "such that"). For example, if we are studying real number solutions of equations, then

$$S = \{x : (x - 1)(x + 2)(x + 3) = 0\}$$

is the set of all real numbers x such that $(x - 1)(x + 2)(x + 3) = 0$; that is, S is the solution set of the equation $(x - 1)(x + 2)(x + 3) = 0$. We could have written $S = \{1, -2, -3\}$; however, even though this way of expressing S is apparently simpler, it does not tell us that we are interested in the solutions of a particular equation. The **absolute value** $|x|$ of a real number x is x if $x \geq 0$; while $|x| = -x$ if $x < 0$. Therefore,

$$T = \{x : |x| = 2\}$$

is the set of all real numbers having absolute value 2, that is, $T = \{2, -2\}$. In the sets S and T that we have just described, we understand that "x" refers to a real number x. If there is a possibility that this wouldn't be clear to the reader, then we should specifically say that x is a real number. We'll say more about this soon. The set

$$P = \{x : x \text{ has been a president of the United States}\}$$

describes, rather obviously, all those individuals who have been president of the United States. So Abraham Lincoln belongs to P but Benjamin Franklin does not.

Example 1.2 *Let $A = \{3, 4, 5, \dots, 20\}$. If B denotes the set consisting of those elements of A that are less than 8, then we can write*

$$B = \{x \in A : x < 8\} = \{3, 4, 5, 6, 7\}.$$ ♦

Some sets are encountered so often that they are given special notation. We use $\mathbf{N}$ to denote the set of all **positive integers** (or **natural numbers**); that is, $\mathbf{N} = \{1, 2, 3, \dots\}$. The set of all **integers** (positive, negative, and zero) is denoted by $\mathbf{Z}$. So $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. With the aid of the notation we've just introduced, we can now describe the set $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even integers by

$$E = \{y : y \text{ is an even integer}\} \text{ or } E = \{2x : x \text{ is an integer}\}, \text{ or as}$$

$$E = \{y : y = 2x \text{ for some } x \in \mathbf{Z}\} \text{ or } E = \{2x : x \in \mathbf{Z}\}.$$

Also,

$$S = \{x^2 : x \text{ is an integer}\} = \{x^2 : x \in \mathbf{Z}\} = \{0, 1, 4, 9, \dots\}$$

describes the set of squares of integers.

The set of **real numbers** is denoted by $\mathbf{R}$, and the set of positive real numbers is denoted by $\mathbf{R}^+$. A real number that can be expressed in the form $\frac{m}{n}$, where $m, n \in \mathbf{Z}$ and $n \neq 0$, is called a **rational number**. For example, $\frac{2}{3}$, $\frac{-5}{11}$, $17 = \frac{17}{1}$ and $\frac{4}{6}$ are rational numbers. Of course, $4/6 = 2/3$. The set of all rational numbers is denoted by $\mathbf{Q}$. A real number that is not rational is called **irrational.** The real numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{2}$, π and e are known to be irrational; that is, none of these numbers can be expressed as the ratio of two integers. It is also known that the real numbers with (infinite) nonrepeating decimal expansions are precisely the irrational numbers. There is no common symbol to denote the set of irrational numbers. We will often use $\mathbf{I}$ for the set of all irrational numbers, however. Thus, $\sqrt{2} \in \mathbf{R}$ and $\sqrt{2} \notin \mathbf{Q}$; so $\sqrt{2} \in \mathbf{I}$.

For a set S, we write $|S|$ to denote the number of elements in S. The number $|S|$ is also referred to as the **cardinal number** or **cardinality** of S. If $A = \{1, 2\}$ and $B = \{1, 2, \{1, 2\}, \emptyset\}$, then $|A| = 2$ and $|B| = 4$. Also, $|\emptyset| = 0$. Although the notation is identical for the cardinality of a set and the absolute value of a real number, we should have no trouble distinguishing between the two. A set S is **finite** if $|S| = n$ for some nonnegative integer n. A set S is **infinite** if it is not finite. For the present, we will use the notation $|S|$ only for finite sets S. In Chapter 10, we will discuss the cardinality of infinite sets.

Let's now consider a few examples of sets that are defined in terms of the special sets we have just described.

Example 1.3 *Let* $D = \{n \in \mathbf{N} : n \le 9\}$, $E = \{x \in \mathbf{Q} : x \le 9\}$, $H = \{x \in \mathbf{R} : x^2 - 2 = 0\}$ *and* $J = \{x \in \mathbf{Q} : x^2 - 2 = 0\}$.

(a) Describe the set D by listing its elements.
(b) Give an example of three elements that belong to E but do not belong to D.
(c) Describe the set H by listing its elements.
(d) Describe the set J in another manner.
(e) Determine the cardinality of each set D, H and J.

Solution

(a) $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
(b) $\frac{7}{5}, 0, -3$.
(c) $H = \{\sqrt{2}, -\sqrt{2}\}$.
(d) $J = \emptyset$.
(e) $|D| = 9$, $|H| = 2$ and $|J| = 0$. ♦

Example 1.4 *In which of the following sets is the integer* -2 *an element?*
$S_1 = \{-1, -2, \{-1\}, \{-2\}, \{-1, -2\}\}$, $S_2 = \{x \in \mathbf{N} : -x \in \mathbf{N}\}$,
$S_3 = \{x \in \mathbf{Z} : x^2 = 2^x\}$, $S_4 = \{x \in \mathbf{Z} : |x| = -x\}$,
$S_5 = \{\{-1, -2\}, \{-2, -3\}, \{-1, -3\}\}$.

Solution The integer -2 is an element of the sets S_1 and S_4. For S_4, $|-2| = 2 = -(-2)$. The set $S_2 = \emptyset$. Since $(-2)^2 = 4$ and $2^{-2} = 1/4$, it follows that $-2 \notin S_3$. Because each element of S_5 is a set, it contains no integers. ♦

A **complex number** is a number of the form $a + bi$, where $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$. A complex number $a + bi$ where $b = 0$, can be expressed as $a + 0i$ or, more simply, as a. Hence $a + 0i = a$ is a real number. Thus every real number is a complex number. Let $\mathbf{C}$ denote the set of complex numbers. If $K = \{x \in \mathbf{C} : x^2 + 1 = 0\}$, then $K = \{i, -i\}$. Of course, if $L = \{x \in \mathbf{R} : x^2 + 1 = 0\}$, then $L = \emptyset$. You might have seen that the sum of two complex numbers $a + bi$ and $c + di$ is $(a + c) + (b + d)i$, while their product is

$$(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

The special sets that we've just described are now summarized below:

symbol	for the set of
N	natural numbers (positive integers)
Z	integers
Q	rational numbers
I	irrational numbers
R	real numbers
C	complex numbers

1.2 Subsets

A set A is called a **subset** of a set B if every element of A also belongs to B. If A is a subset of B, then we write $A \subseteq B$. If A, B and C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. To see why this is so, suppose that some element x belongs to A. Because $A \subseteq B$, it follows that $x \in B$. But $B \subseteq C$, which implies that $x \in C$. Therefore, every element that belongs to A also belongs to C and so $A \subseteq C$. This property of subsets might remind you of the property of real numbers where if $a, b, c \in \mathbf{R}$ such that if $a \leq b$ and $b \leq c$, then $a \leq c$. For the sets $X = \{1, 3, 6\}$ and $Y = \{1, 2, 3, 5, 6\}$, we have $X \subseteq Y$. Also, $\mathbf{N} \subseteq \mathbf{Z}$ and $\mathbf{Q} \subseteq \mathbf{R}$. In addition, $\mathbf{R} \subseteq \mathbf{C}$. Since $\mathbf{Q} \subseteq \mathbf{R}$ and $\mathbf{R} \subseteq \mathbf{C}$, it therefore follows that $\mathbf{Q} \subseteq \mathbf{C}$. Moreover, every set is a subset of itself.

Example 1.5 *Find two sets A and B such that A is both an element of and a subset of B.*

Solution Suppose that we seek two sets A and B such that $A \in B$ and $A \subseteq B$. Let's start with a simple example for A, say $A = \{1\}$. Since we want $A \in B$, the set B must contain the set $\{1\}$ as one of its elements. On the other hand, we also require that $A \subseteq B$, so every element of A must belong to B. Since 1 is the only element of A, it follows that B must also contain the number 1. A possible choice for B is then $B = \{1, \{1\}\}$, although $B = \{1, 2, \{1\}\}$ would also satisfy the conditions. ♦

In the following example, we will see how we arrive at the answer to a question asked there. This is a prelude to logic, which will be discussed in Chapter 2.

Example 1.6 *Two sets A and B have the property that each is a subset of* $\{1, 2, 3, 4, 5\}$ *and* $|A| = |B| = 3$. *Furthermore,*

(a) 1 *belongs to A but not to B.*
(b) 2 *belongs to B but not to A.*
(c) 3 *belongs to exactly one of A and B.*
(d) 4 *belongs to exactly one of A and B.*
(e) 5 *belongs to at least one of A and B.*

What are the possibilities for the set A?

Solution By (a) and (b), $1 \in A$ and $1 \notin B$, while $2 \in B$ and $2 \notin A$. By (c), 3 belongs to A or B but not both. By (d), 4 belongs to A or B but not both. If 3 and 4 belong to the same set, then either 3 and 4 both belong to A or 3 and 4 both belong to B. Should it occur that $3 \in A$ and $4 \in A$, then $1 \notin B$, $3 \notin B$ and $4 \notin B$. This means that $|B| \neq 3$. On the other hand, if $3 \in B$ and $4 \in B$, then $3 \notin A$ and $4 \notin A$. Therefore, A contains none of 2, 3 and 4 and so $|A| \neq 3$. We can therefore conclude that 3 and 4 belong to different sets. The only way that $|A| = |B| = 3$ is for 5 to belong to both A and B and so either $A = \{1, 3, 5\}$ or $A = \{1, 4, 5\}$. ♦

If a set C is *not* a subset of a set D, then we write $C \nsubseteq D$. In this case, there must be some element of C that is not an element of D. One consequence of this is that the empty set $\emptyset$ is a subset of every set. If this were not the case, then there must be some set A such that $\emptyset \not\subseteq A$. But this would mean there is some element, say x, in $\emptyset$ that is not in A. However, $\emptyset$ contains no elements. So $\emptyset \subseteq A$ for *every* set A.

Example 1.7 *Let* $S = \{1, \{2\}, \{1, 2\}\}$.

(a) Determine which of the following are elements of S:
$1, \{1\}, 2, \{2\}, \{1, 2\}, \{\{1, 2\}\}$.

(b) Determine which of the following are subsets of S:
$\{1\}, \{2\}, \{1, 2\}, \{\{1\}, 2\}, \{1, \{2\}\}, \{\{1\}, \{2\}\}, \{\{1, 2\}\}$.

Solution

(a) The following are elements of S: $1, \{2\}, \{1, 2\}$.

(b) The following are subsets of S: $\{1\}, \{1, \{2\}\}, \{\{1, 2\}\}$. ♦

In a typical discussion of sets, we are ordinarily concerned with subsets of some specified set U, called the **universal set.** For example, we may be dealing only with integers, in which case the universal set is **Z**, or we may be dealing only with real numbers, in which case the universal set is **R**. On the other hand, the universal set being considered may be neither **Z** nor **R**. Indeed, U may not even be a set of numbers.

Some frequently encountered subsets of **R** are the so-called "intervals," which you have no doubt encountered often. For $a, b \in \mathbf{R}$ and $a < b$, the **open interval** (a, b) is the set

$$(a, b) = \{x \in \mathbf{R} : \ a < x < b\}.$$

Therefore, all of the real numbers $\frac{5}{2}, \sqrt{5}, e, 3, \pi, 4.99$ belong to $(2, 5)$, but none of the real numbers $\sqrt{2}, 1.99, 2, 5$ belong to $(2, 5)$.

For $a, b \in \mathbf{R}$ and $a \leq b$, the **closed interval** $[a, b]$ is the set

$$[a, b] = \{x \in \mathbf{R} : \ a \leq x \leq b\}.$$

While $2, 5 \notin (2, 5)$, we do have $2, 5 \in [2, 5]$. The "interval" $[a, a]$ is therefore $\{a\}$. Thus, for $a < b$, we have $(a, b) \subseteq [a, b]$. For $a, b \in \mathbf{R}$ and $a < b$, the **half-open** or **half-closed intervals** $[a, b)$ and $(a, b]$ are defined as expected:

$$[a, b) = \{x \in \mathbf{R} : \ a \leq x < b\} \text{ and } (a, b] = \{x \in \mathbf{R} : \ a < x \leq b\}.$$

For $a \in \mathbf{R}$, the infinite intervals $(-\infty, a)$, $(-\infty, a]$, (a, ∞) and $[a, \infty)$ are defined as

$$(-\infty, a) = \{x \in \mathbf{R} : \ x < a\}, \ (-\infty, a] = \{x \in \mathbf{R} : \ x \leq a\},$$
$$(a, \infty) = \{x \in \mathbf{R} : \ x > a\}, \quad [a, \infty) = \{x \in \mathbf{R} : \ x \geq a\}.$$

The interval $(-\infty, \infty)$ is the set **R**. Note that the infinity symbols ∞ and $-\infty$ are not real numbers; they are only used to help describe certain intervals. Therefore, $[1, \infty]$, for example, has no meaning.

Two sets A and B are **equal**, indicated by writing $A = B$, if they have exactly the same elements. Another way of saying $A = B$ is that every element of A is in B and every element of B is in A, that is, $A \subseteq B$ and $B \subseteq A$. In particular, whenever some element x belongs to A, then $x \in B$ because $A \subseteq B$. Also, if y is an element of B, then because $B \subseteq A$, it follows that $y \in A$. That is, whenever an element belongs to one of these sets, it must belong to the other and so $A = B$. This fact will be very useful to us

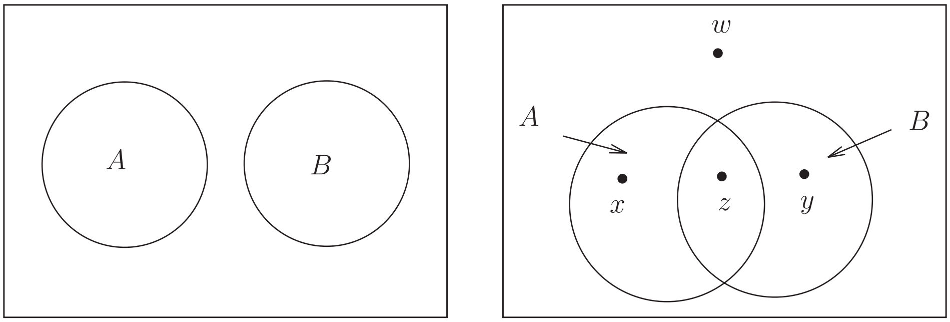

Figure 1.1 Venn diagrams for two sets A and B

in Chapter 4. If $A \neq B$, then there must be some element belonging to one of A and B but not to the other.

It is often convenient to represent sets by diagrams called **Venn diagrams.** For example, Figure 1.1 shows Venn diagrams for two sets A and B. The diagram on the left represents two sets A and B that have no elements in common, while the diagram on the right is more general. The element x belongs to A but not to B, the element y belongs to B but not to A, the element z belongs to both A and B, while w belongs to neither A nor B. In general, the elements of a set are understood to be those displayed within the region that describes the set. A rectangle in a Venn diagram represents the universal set in this case. Since every element under consideration belongs to the universal set, each element in a Venn diagram lies within the rectangle.

A set A is a **proper subset** of a set B if $A \subseteq B$ but $A \neq B$. If A is a proper subset of B, then we write $A \subset B$. For example, if $S = \{4, 5, 7\}$ and $T = \{3, 4, 5, 6, 7\}$, then $S \subset T$. (Although we write $A \subset B$ to indicate that A is a proper subset of B, it should be mentioned that some prefer to write $A \subsetneq B$ to indicate that A is a proper subset of B. Indeed, there are some who write $A \subset B$, rather than $A \subseteq B$, to indicate that A is a subset of B. We will follow the notation introduced above, however.)

The set consisting of all subsets of a given set A is called the **power set** of A and is denoted by $\mathcal{P}(A)$.

Example 1.8 *For each set A below, determine $\mathcal{P}(A)$. In each case, determine $|A|$ and $|\mathcal{P}(A)|$.*

(a) $A = \emptyset$, (b) $A = \{a, b\}$, (c) $A = \{1, 2, 3\}$.

Solution

(a) $\mathcal{P}(A) = \{\emptyset\}$. In this case, $|A| = 0$ and $|\mathcal{P}(A)| = 1$.

(b) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this case, $|A| = 2$ and $|\mathcal{P}(A)| = 4$.

(c) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
In this case, $|A| = 3$ and $|\mathcal{P}(A)| = 8$. ♦

Notice that for each set A in Example 1.8, we have $|\mathcal{P}(A)| = 2^{|A|}$. In fact, if A is any finite set, with n elements say, then $\mathcal{P}(A)$ has 2^n elements; that is,

$$|\mathcal{P}(A)| = 2^{|A|}$$

for every finite set A. (Later we will explain why this is true.)

Example 1.9 *If* $C = \{\emptyset, \{\emptyset\}\}$*, then*

$$\mathcal{P}(C) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

It is important to note that no two of the sets $\emptyset$, $\{\emptyset\}$ *and* $\{\{\emptyset\}\}$ *are equal. (An empty box and a box containing an empty box are not the same.) For the set C above, it is therefore correct to write*

$$\emptyset \subseteq C, \emptyset \subset C, \emptyset \in C, \{\emptyset\} \subseteq C, \{\emptyset\} \subset C, \{\emptyset\} \in C,$$

as well as

$$\{\{\emptyset\}\} \subseteq C, \{\{\emptyset\}\} \notin C, \{\{\emptyset\}\} \in \mathcal{P}(C).$$ ♦

1.3 Set Operations

Just as there are several ways of combining two integers to produce another integer (addition, subtraction, multiplication and sometimes division), there are several ways to combine two sets to produce another set. The **union** of two sets A and B, denoted by $A \cup B$, is the set of all elements belonging to A or B, that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The use of the word "or" here, and in mathematics in general, allows an element of $A \cup B$ to belong to both A and B. That is, x is in $A \cup B$ if x is in A or x is in B or x is in both A and B. A Venn diagram for $A \cup B$ is shown in Figure 1.2.

Example 1.10 *For the sets* $A_1 = \{2, 5, 7, 8\}$*,* $A_2 = \{1, 3, 5\}$ *and* $A_3 = \{2, 4, 6, 8\}$*, we have*

$$A_1 \cup A_2 = \{1, 2, 3, 5, 7, 8\},$$
$$A_1 \cup A_3 = \{2, 4, 5, 6, 7, 8\},$$
$$A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 8\}.$$

Also, $\mathbf{N} \cup \mathbf{Z} = \mathbf{Z}$ *and* $\mathbf{Q} \cup \mathbf{I} = \mathbf{R}$. ♦

The **intersection** of two sets A and B is the set of all elements belonging to both A and B. The intersection of A and B is denoted by $A \cap B$. In symbols,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A Venn diagram for $A \cap B$ is shown in Figure 1.3.

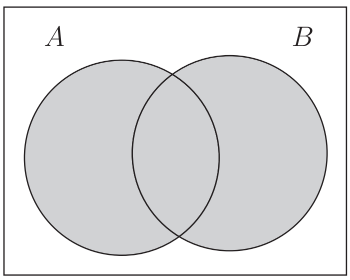

Figure 1.2 A Venn diagram for $A \cup B$

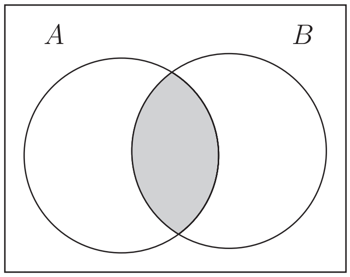

Figure 1.3 A Venn diagram for $A \cap B$

Example 1.11 *For the sets A_1, A_2 and A_3 described in Example 1.10,*

$$A_1 \cap A_2 = \{5\},\ A_1 \cap A_3 = \{2, 8\} \text{ and } A_2 \cap A_3 = \emptyset.$$

Also, $\mathbf{N} \cap \mathbf{Z} = \mathbf{N}$ *and* $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$. ♦

For every two sets A and B, it follows that

$$A \cap B \subseteq A \cup B.$$

To see why this is true, suppose that x is an element belonging to $A \cap B$. Then x belongs to both A and B. Since $x \in A$, for example, $x \in A \cup B$ and so $A \cap B \subseteq A \cup B$.

If two sets A and B have no elements in common, then $A \cap B = \emptyset$ and A and B are said to be **disjoint**. Consequently, the sets A_2 and A_3 described in Example 1.10 are disjoint; however, A_1 and A_3 are not disjoint since 2 and 8 belong to both sets. Also, $\mathbf{Q}$ and $\mathbf{I}$ are disjoint.

The **difference** $A - B$ of two sets A and B (also written as $A \setminus B$ by some mathematicians) is defined as

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

A Venn diagram for $A - B$ is shown in Figure 1.4.

Example 1.12 *For the sets $A_1 = \{2, 5, 7, 8\}$ and $A_2 = \{1, 3, 5\}$ in Examples 1.10 and 1.11,* $A_1 - A_2 = \{2, 7, 8\}$ *and* $A_2 - A_1 = \{1, 3\}$. *Furthermore,* $\mathbf{R} - \mathbf{Q} = \mathbf{I}$. ♦

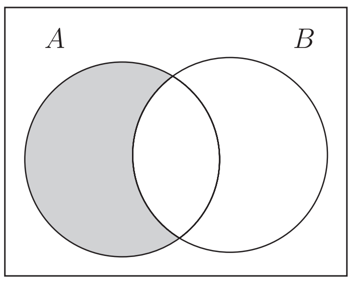

Figure 1.4 A Venn diagram for $A - B$

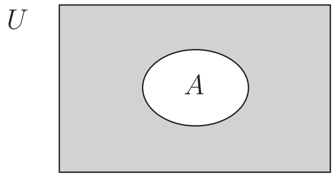

Figure 1.5 A Venn diagram for $\overline{A}$

Example 1.13 *For* $A = \{x \in \mathbf{R} : |x| \le 3\}$, $B = \{x \in \mathbf{R} : |x| > 2\}$ *and* $C = \{x \in \mathbf{R} : |x - 1| \le 4\}$:

(a) Express A, B *and* C *using interval notation.*

(b) Determine $A \cap B$, $A - B$, $B \cap C$, $B \cup C$, $B - C$ *and* $C - B$.

Solution

(a) $A = [-3, 3]$, $B = (-\infty, -2) \cup (2, \infty)$ and $C = [-3, 5]$.

(b) $A \cap B = [-3, -2) \cup (2, 3]$, $A - B = [-2, 2]$, $B \cap C = [-3, -2) \cup (2, 5]$, $B \cup C = (-\infty, \infty)$, $B - C = (-\infty, -3) \cup (5, \infty)$ and $C - B = [-2, 2]$. ♦

Suppose that we are considering a certain universal set U, that is, all sets being discussed are subsets of U. For a set A, its **complement** is

$$\overline{A} = U - A = \{x : x \in U \text{ and } x \notin A\}.$$

If $U = \mathbf{Z}$, then $\overline{\mathbf{N}} = \{0, -1, -2, \ldots\}$; while if $U = \mathbf{R}$, then $\overline{\mathbf{Q}} = \mathbf{I}$. A Venn diagram for $\overline{A}$ is shown in Figure 1.5.

The set difference $A - B$ is sometimes called the **relative complement** of B in A. Indeed, from the definition, $A - B = \{x : x \in A \text{ and } x \notin B\}$. The set $A - B$ can also be expressed in terms of complements, namely, $A - B = A \cap \overline{B}$. This fact will be established later.

Example 1.14 *Let* $U = \{1, 2, \ldots, 10\}$ *be the universal set,* $A = \{2, 3, 5, 7\}$ *and* $B = \{2, 4, 6, 8, 10\}$. *Determine each of the following*:

(a) $\overline{B}$, (b) $A - B$, (c) $A \cap \overline{B}$, (d) $\overline{\overline{B}}$.

Solution

(a) $\overline{B} = \{1, 3, 5, 7, 9\}$.

(b) $A - B = \{3, 5, 7\}$.

(c) $A \cap \overline{B} = \{3, 5, 7\} = A - B$.

(d) $\overline{\overline{B}} = B = \{2, 4, 6, 8, 10\}$. ♦

Example 1.15 *Let* $A = \{0, \{0\}, \{0, \{0\}\}\}$.

(a) Determine which of the following are elements of A: 0, $\{0\}$, $\{\{0\}\}$.

(b) Determine $|A|$.

(c) Determine which of the following are subsets of A: 0, $\{0\}$, $\{\{0\}\}$.
For (d)–(i), determine the indicated sets.

(d) $\{0\} \cap A$

(e) $\{\{0\}\} \cap A$

(f) $\{\{\{0\}\}\} \cap A$

(g) $\{0\} \cup A$

(h) $\{\{0\}\} \cup A$

(i) $\{\{\{0\}\}\} \cup A$.

Solution

(a) While 0 and $\{0\}$ are elements of A, $\{\{0\}\}$ is not an element of A.

(b) The set A has three elements: 0, $\{0\}$, $\{0, \{0\}\}$. Therefore, $|A| = 3$.

(c) The integer 0 is not a set and so cannot be a subset of A (or a subset of any other set). Since $0 \in A$ and $\{0\} \in A$, it follows that $\{0\} \subseteq A$ and $\{\{0\}\} \subseteq A$.

(d) Since 0 is the only element that belongs to both $\{0\}$ and A, it follows that $\{0\} \cap A = \{0\}$.

(e) Since $\{0\}$ is the only element that belongs to both $\{\{0\}\}$ and A, it follows that $\{\{0\}\} \cap A = \{\{0\}\}$.

(f) Since $\{\{0\}\}$ is not an element of A, it follows that $\{\{\{0\}\}\}$ and A are disjoint sets and so $\{\{\{0\}\}\} \cap A = \emptyset$.

(g) Since $0 \in A$, it follows that $\{0\} \cup A = A$.

(h) Since $\{0\} \in A$, it follows that $\{\{0\}\} \cup A = A$.

(i) Since $\{\{0\}\} \notin A$, it follows that $\{\{\{0\}\}\} \cup A = \{0, \{0\}, \{\{0\}\}, \{0, \{0\}\}\}$. ♦

1.4 Indexed Collections of Sets

We will often encounter situations where more than two sets are combined using the set operations we described earlier. In the case of three sets A, B and C, the standard Venn diagram is shown in Figure 1.6.

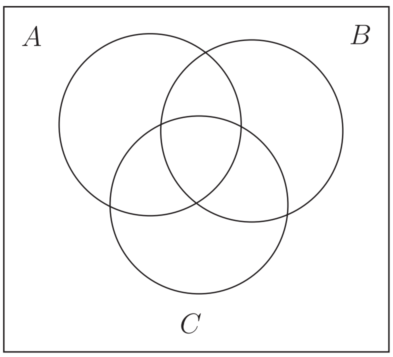

Figure 1.6 A Venn diagram for three sets

The union $A \cup B \cup C$ is defined as

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}.$$

Thus, in order for an element to belong to $A \cup B \cup C$, the element must belong to at least one of the sets A, B and C. Because it is often useful to consider the union of several sets, additional notation is needed. The union of the $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \cup A_2 \cup \cdots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$ and is defined as

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i, 1 \leq i \leq n\}.$$

Thus, for an element a to belong to $\bigcup_{i=1}^{n} A_i$, it is necessary that a belongs to at least one of the sets $A_1, A_2, \ldots, A_n$.

Example 1.16 *Let $B_1 = \{1, 2\}$, $B_2 = \{2, 3\}, \ldots, B_{10} = \{10, 11\}$; that is, $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine each of the following:*

(a) $\bigcup_{i=1}^{5} B_i$. (b) $\bigcup_{i=1}^{10} B_i$. (c) $\bigcup_{i=3}^{7} B_i$. (d) $\bigcup_{i=j}^{k} B_i$, where $1 \leq j \leq k \leq 10$.

Solution (a) $\bigcup_{i=1}^{5} B_i = \{1, 2, \ldots, 6\}$. (b) $\bigcup_{i=1}^{10} B_i = \{1, 2, \ldots, 11\}$

(c) $\bigcup_{i=3}^{7} B_i = \{3, 4, \ldots, 8\}$. (d) $\bigcup_{i=j}^{k} B_i = \{j, j+1, \ldots, k+1\}$. ♦

We are often interested in the intersection of several sets as well. The intersection of the $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is expressed as $A_1 \cap A_2 \cap \cdots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$ and is defined by

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for every } i, 1 \leq i \leq n\}.$$

The next example concerns the sets mentioned in Example 1.16.

Example 1.17 *Let $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine the following:*

(a) $\bigcap_{i=1}^{10} B_i$. (b) $B_i \cap B_{i+1}$. (c) $\bigcap_{i=j}^{j+1} B_i$, where $1 \leq j < 10$.

(d) $\bigcap_{i=j}^{k} B_i$ where $1 \leq j < k \leq 10$.

Solution (a) $\bigcap_{i=1}^{10} B_i = \emptyset$. (b) $B_i \cap B_{i+1} = \{i+1\}$. (c) $\bigcap_{i=j}^{j+1} B_i = \{j+1\}$.

(d) $\bigcap_{i=j}^{k} B_i = \{j+1\}$ if $k = j+1$; while $\bigcap_{i=j}^{k} B_i = \emptyset$ if $k > j+1$. ♦

There are instances when the union or intersection of a collection of sets cannot be described conveniently (or perhaps at all) in the manner mentioned above. For this reason, we introduce a (nonempty) set I, called an **index set**, which is used as a mechanism for selecting those sets we want to consider. For example, for an index set I, suppose that there is a set S_α for each $\alpha \in I$. We write $\{S_\alpha\}_{\alpha \in I}$ to describe the collection of all sets S_α, where $\alpha \in I$. Such a collection is called an **indexed collection of sets**. We define the union of the sets in $\{S_\alpha\}_{\alpha \in I}$ by

$$\bigcup_{\alpha \in I} S_\alpha = \{x : x \in S_\alpha \text{ for some } \alpha \in I\},$$

and the intersection of these sets by

$$\bigcap_{\alpha \in I} S_\alpha = \{x : x \in S_\alpha \text{ for all } \alpha \in I\}.$$

Hence an element a belongs to $\bigcup_{\alpha \in I} S_\alpha$ if a belongs to at least one of the sets in the collection $\{S_\alpha\}_{\alpha \in I}$, while a belongs to $\bigcap_{\alpha \in I} S_\alpha$ if a belongs to every set in the collection $\{S_\alpha\}_{\alpha \in I}$. We refer to $\bigcup_{\alpha \in I} S_\alpha$ as the union of the collection $\{S_\alpha\}_{\alpha \in I}$ and $\bigcap_{\alpha \in I} S_\alpha$ as the intersection of the collection $\{S_\alpha\}_{\alpha \in I}$. Just as there is nothing special about our choice of i in $\bigcup_{i=1}^{n} A_i$ (that is, we could just as well describe this set by $\bigcup_{j=1}^{n} A_j$, say), there is nothing special about α in $\bigcup_{\alpha \in I} S_\alpha$. We could also describe this set by $\bigcup_{x \in I} S_x$. The variables i and α above are *dummy variables* and any appropriate symbol could be used. Indeed, we could write J or some other symbol for an index set.

Example 1.18 *For $n \in \mathbf{N}$, define $S_n = \{n, 2n\}$. For example, $S_1 = \{1, 2\}$, $S_2 = \{2, 4\}$ and $S_4 = \{4, 8\}$. Then $S_1 \cup S_2 \cup S_4 = \{1, 2, 4, 8\}$. We can also describe this set by means of an index set. If we let $I = \{1, 2, 4\}$, then*

$$\bigcup_{\alpha \in I} S_\alpha = S_1 \cup S_2 \cup S_4.$$ ♦

Example 1.19 *For each $n \in \mathbf{N}$, define A_n to be the closed interval $[-\frac{1}{n}, \frac{1}{n}]$ of real numbers; that is,*

$$A_n = \left\{x \in \mathbf{R} : -\frac{1}{n} \leq x \leq \frac{1}{n}\right\}.$$

So $A_1 = [-1, 1]$, $A_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $A_3 = \left[-\frac{1}{3}, \frac{1}{3}\right]$ and so on. We have now defined the sets $A_1, A_2, A_3, \ldots$. The union of these sets can be written as $A_1 \cup A_2 \cup A_3 \cup \cdots$ or $\bigcup_{i=1}^{\infty} A_i$. Using $\mathbf{N}$ as an index set, we can also write this union as $\bigcup_{n \in \mathbf{N}} A_n$. Since $A_n \subseteq A_1 = [-1, 1]$ for every $n \in \mathbf{N}$, it follows that $\bigcup_{n \in \mathbf{N}} A_n = [-1, 1]$. Certainly, $0 \in A_n$ for every $n \in \mathbf{N}$; in fact, $\bigcap_{n \in \mathbf{N}} A_n = \{0\}$. ♦

Example 1.20 *Let A denote the set of the letters of the alphabet, that is, $A = \{a, b, \ldots, z\}$. For $\alpha \in A$, let A_α consist of α and the two letters that follow α. So $A_a = \{a, b, c\}$ and $A_b = \{b, c, d\}$. By A_y, we will mean the set $\{y, z, a\}$ and $A_z = \{z, a, b\}$. Hence $|A_\alpha| = 3$ for every $\alpha \in A$. Therefore $\bigcup_{\alpha \in A} A_\alpha = A$. Indeed, if*

$$B = \{a, d, g, j, m, p, s, v, y\},$$

then $\bigcup_{\alpha \in B} A_\alpha = A$ *as well. On the other hand, if* $I = \{p, q, r\}$*, then* $\bigcup_{\alpha \in I} A_\alpha = \{p, q, r, s, t\}$ *while* $\bigcap_{\alpha \in I} A_\alpha = \{r\}$*.* ♦

Example 1.21 *Let* $S = \{1, 2, \ldots, 10\}$*. Each of the sets*

$$S_1 = \{1, 2, 3, 4\},\ S_2 = \{4, 5, 6, 7, 8\} \text{ and } S_3 = \{7, 8, 9, 10\}$$

is a subset of S*. Also,* $S_1 \cup S_2 \cup S_3 = S$*. This union can be described in a number of ways. Define* $I = \{1, 2, 3\}$ *and* $J = \{S_1, S_2, S_3\}$*. Then the union of the three sets belonging to* J *is precisely* $S_1 \cup S_2 \cup S_3$*, which can also be written as*

$$S = S_1 \cup S_2 \cup S_3 = \bigcup_{i=1}^{3} S_i = \bigcup_{\alpha \in I} S_\alpha = \bigcup_{X \in J} X.$$ ♦

1.5 Partitions of Sets

Recall that two sets are disjoint if their intersection is the empty set. A collection $\mathcal{S}$ of subsets of a set A is called **pairwise disjoint** if every two distinct subsets that belong to $\mathcal{S}$ are disjoint. For example, let $A = \{1, 2, \ldots, 7\}$, $B = \{1, 6\}$, $C = \{2, 5\}$, $D = \{4, 7\}$ and $S = \{B, C, D\}$. Then S is a pairwise disjoint collection of subsets of A since $B \cap C = B \cap D = C \cap D = \emptyset$. On the other hand, let $A' = \{1, 2, 3\}$, $B' = \{1, 2\}$, $C' = \{1, 3\}$, $D' = \{2, 3\}$ and $S' = \{B', C', D'\}$. Although S' is a collection of subsets of A' and $B' \cap C' \cap D' = \emptyset$, the set S' is *not* a pairwise disjoint collection of sets since $B' \cap C' \neq \emptyset$, for example. Indeed, $B' \cap D'$ and $C' \cap D'$ are also nonempty.

We will often have the occasion (especially in Chapter 8) to encounter, for a nonempty set A, a collection $\mathcal{S}$ of pairwise disjoint nonempty subsets of A with the added property that every element of A belongs to some subset in $\mathcal{S}$. Such a collection is called a **partition** of A. A **partition** of A can also be defined as a collection $\mathcal{S}$ of nonempty subsets of A such that every element of A belongs to exactly one subset in $\mathcal{S}$. Furthermore, a partition of A can be defined as a collection $\mathcal{S}$ of subsets of A satisfying the three properties:

(1) $X \neq \emptyset$ for every set $X \in \mathcal{S}$;
(2) for every two sets $X, Y \in \mathcal{S}$, either $X = Y$ or $X \cap Y = \emptyset$;
(3) $\bigcup_{X \in \mathcal{S}} X = A$.

Example 1.22 *Consider the following collections of subsets of the set* $A = \{1, 2, 3, 4, 5, 6\}$:

$$S_1 = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\};$$
$$S_2 = \{\{1, 2, 3\}, \{4\}, \emptyset, \{5, 6\}\};$$
$$S_3 = \{\{1, 2\}, \{3, 4, 5\}, \{5, 6\}\};$$
$$S_4 = \{\{1, 4\}, \{3, 5\}, \{2\}\}.$$

Determine which of these sets are partitions of A*.*

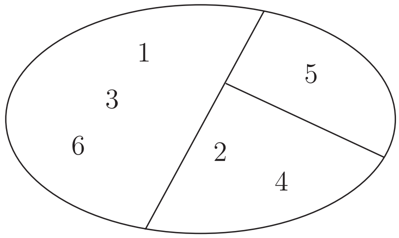

Figure 1.7 A partition of a set

Solution The set S_1 is a partition of A. The set S_2 is not a partition of A since $\emptyset$ is one of the elements of S_2. The set S_3 is not a partition of A either since the element 5 belongs to two distinct subsets in S_3, namely, $\{3, 4, 5\}$ and $\{5, 6\}$. Finally, S_4 is also not a partition of A because the element 6 belongs to no subset in S_4. ♦

As the word *partition* probably suggests, a partition of a nonempty set A is a division of A into nonempty subsets. The partition S_1 of the set A in Example 1.22 is illustrated in the diagram shown in Figure 1.7.

For example, the set **Z** of integers can be partitioned into the set of even integers and the set of odd integers. The set **R** of real numbers can be partitioned into the set $\mathbf{R}^+$ of positive real numbers, the set of negative real numbers and the set $\{0\}$ consisting of the number 0. In addition, **R** can be partitioned into the set **Q** of rational numbers and the set **I** of irrational numbers.

Example 1.23 *Let* $A = \{1, 2, \dots, 12\}$.

(a) Give an example of a partition S of A such that $|S| = 5$.

(b) Give an example of a subset T of the partition S in (a) such that $|T| = 3$.

(c) List all those elements B in the partition S in (a) such that $|B| = 2$.

Solution

(a) We are seeking a partition S of A consisting of five subsets. One such example is

$$S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

(b) We are seeking a subset T of S (given in (a)) consisting of three elements. One such example is

$$T = \{\{1, 2\}, \{3, 4\}, \{7, 8, 9\}\}.$$

(c) We have been asked to list all those elements of S (given in (a)) consisting of two elements of A. These elements are: $\{1, 2\}, \{3, 4\}, \{5, 6\}$. ♦

1.6 Cartesian Products of Sets

We've already mentioned that when a set A is described by listing its elements, the order in which the elements of A are listed doesn't matter. That is, if the set A consists of two elements x and y, then $A = \{x, y\} = \{y, x\}$. When we speak of the **ordered pair** (x, y), however, this is another story. The ordered pair (x, y) is a single element consisting of a pair of elements in which x is the first element (or first coordinate) of the ordered pair (x, y) and y is the second element (or second coordinate). Moreover, for two ordered pairs (x, y) and (w, z) to be equal, that is, $(x, y) = (w, z)$, we must have $x = w$ and $y = z$. So, if $x \neq y$, then $(x, y) \neq (y, x)$.

The **Cartesian product** (or simply the product) $A \times B$ of two sets A and B is the set consisting of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. In other words,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example 1.24 *If $A = \{x, y\}$ and $B = \{1, 2, 3\}$, then*

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\},$$

while

$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

Since, for example, $(x, 1) \in A \times B$ and $(x, 1) \notin B \times A$, these two sets do not contain the same elements; so $A \times B \neq B \times A$. Also,

$$A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$$

and

$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}. \quad \blacklozenge$$

We also note that if $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

The Cartesian product $\mathbf{R} \times \mathbf{R}$ is the set of all points in the Euclidean plane. For example, the graph of the straight line $y = 2x + 3$ is the set

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} : y = 2x + 3\}.$$

For the sets $A = \{x, y\}$ and $B = \{1, 2, 3\}$ given in Example 1.24, $|A| = 2$ and $|B| = 3$, while $|A \times B| = 6$. Indeed, for all finite sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

Cartesian products will be explored in more detail in Chapter 7.

EXERCISES FOR CHAPTER 1

Section 1.1: Describing a Set

1.1. Which of the following are sets?

(a) $1, 2, 3$
(b) $\{1, 2\}, 3$
(c) $\{\{1\}, 2\}, 3$
(d) $\{1, \{2\}, 3\}$
(e) $\{1, 2, a, b\}$.

1.2. Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x.

(a) $A = \{1, 2, 3\}$
(b) $B = \{0, 1, 2, 3\}$
(c) $C = \{-2, -1\}$
(d) $D = \{-2, 2, 3\}$

1.3. Determine the cardinality of each of the following sets:

(a) $A = \{1, 2, 3, 4, 5\}$
(b) $B = \{0, 2, 4, \ldots, 20\}$
(c) $C = \{25, 26, 27, \ldots, 75\}$
(d) $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$
(e) $E = \{\emptyset\}$
(f) $F = \{2, \{2, 3, 4\}\}$

1.4. Write each of the following sets by listing its elements within braces.

(a) $A = \{n \in \mathbf{Z} : -4 < n \le 4\}$
(b) $B = \{n \in \mathbf{Z} : n^2 < 5\}$
(c) $C = \{n \in \mathbf{N} : n^3 < 100\}$
(d) $D = \{x \in \mathbf{R} : x^2 - x = 0\}$
(e) $E = \{x \in \mathbf{R} : x^2 + 1 = 0\}$

1.5. Write each of the following sets in the form $\{x \in \mathbf{Z} : p(x)\}$, where $p(x)$ is a property concerning x.

(a) $A = \{-1, -2, -3, \ldots\}$
(b) $B = \{-3, -2, \ldots, 3\}$
(c) $C = \{-2, -1, 1, 2\}$

1.6. The set $E = \{2x : x \in \mathbf{Z}\}$ can be described by listing its elements, namely $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. List the elements of the following sets in a similar manner.

(a) $A = \{2x + 1 : x \in \mathbf{Z}\}$
(b) $B = \{4n : n \in \mathbf{Z}\}$
(c) $C = \{3q + 1 : q \in \mathbf{Z}\}$

1.7. The set $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbf{Z}\} = \{2x : x \in \mathbf{Z}\}$. Describe the following sets in a similar manner.

(a) $A = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$
(b) $B = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$
(c) $C = \{1, 8, 27, 64, 125, \ldots\}$

1.8. Let $A = \{n \in \mathbf{Z} : 2 \le |n| < 4\}$, $B = \{x \in \mathbf{Q} : 2 < x \le 4\}$,
$C = \{x \in \mathbf{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$ and $D = \{x \in \mathbf{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$.

(a) Describe the set A by listing its elements.
(b) Give an example of three elements that belong to B but do not belong to A.
(c) Describe the set C by listing its elements.
(d) Describe the set D in another manner.
(e) Determine the cardinality of each of the sets A, C and D.

1.9. For $A = \{2, 3, 5, 7, 8, 10, 13\}$, let

$$B = \{x \in A : x = y + z, \text{ where } y, z \in A\} \text{ and } C = \{r \in B : r + s \in B \text{ for some } s \in B\}.$$

Determine C.

Section 1.2: Subsets

1.10. Give examples of three sets A, B and C such that

(a) $A \subseteq B \subset C$
(b) $A \in B$, $B \in C$ and $A \notin C$
(c) $A \in B$ and $A \subset C$.

1.11. Let (a, b) be an open interval of real numbers and let $c \in (a, b)$. Describe an open interval I centered at c such that $I \subseteq (a, b)$.

1.12. Which of the following sets are equal?

$A = \{n \in \mathbf{Z} : |n| < 2\}$ $D = \{n \in \mathbf{Z} : n^2 \leq 1\}$
$B = \{n \in \mathbf{Z} : n^3 = n\}$ $E = \{-1, 0, 1\}$.
$C = \{n \in \mathbf{Z} : n^2 \leq n\}$

1.13. For a universal set $U = \{1, 2, \dots, 8\}$ and two sets $A = \{1, 3, 4, 7\}$ and $B = \{4, 5, 8\}$, draw a Venn diagram that represents these sets.

1.14. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for

(a) $A = \{1, 2\}$.
(b) $A = \{\emptyset, 1, \{a\}\}$.

1.15. Find $\mathcal{P}(A)$ for $A = \{0, \{0\}\}$.

1.16. Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

1.17. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{0, \emptyset, \{\emptyset\}\}$.

1.18. For $A = \{x : x = 0 \text{ or } x \in \mathcal{P}(\{0\})\}$, determine $\mathcal{P}(A)$.

1.19. Give an example of a set S such that

(a) $S \subseteq \mathcal{P}(\mathbf{N})$
(b) $S \in \mathcal{P}(\mathbf{N})$
(c) $S \subseteq \mathcal{P}(\mathbf{N})$ and $|S| = 5$
(d) $S \in \mathcal{P}(\mathbf{N})$ and $|S| = 5$

1.20. Determine whether the following statements are true or false.

(a) If $\{1\} \in \mathcal{P}(A)$, then $1 \in A$ but $\{1\} \notin A$.
(b) If A, B and C are sets such that $A \subset \mathcal{P}(B) \subset C$ and $|A| = 2$, then $|C|$ can be 5 but $|C|$ cannot be 4.
(c) If a set B has one more element than a set A, then $\mathcal{P}(B)$ has at least two more elements than $\mathcal{P}(A)$.
(d) If four sets A, B, C and D are subsets of $\{1, 2, 3\}$ such that $|A| = |B| = |C| = |D| = 2$, then at least two of these sets are equal.

1.21. Three subsets A, B and C of $\{1, 2, 3, 4, 5\}$ have the same cardinality. Furthermore,

(a) 1 belongs to A and B but not to C.
(b) 2 belongs to A and C but not to B.
(c) 3 belongs to A and exactly one of B and C.
(d) 4 belongs to an even number of A, B and C.

(e) 5 belongs to an odd number of A, B and C.
(f) The sums of the elements in two of the sets A, B and C differ by 1.

What is B?

Section 1.3: Set Operations

1.22. Let $U = \{1, 3, \ldots, 15\}$ be the universal set, $A = \{1, 5, 9, 13\}$, and $B = \{3, 9, 15\}$. Determine the following:

(a) $A \cup B$ (b) $A \cap B$ (c) $A - B$ (d) $B - A$ (e) $\overline{A}$ (f) $A \cap \overline{B}$.

1.23. Give examples of two sets A and B such that $|A - B| = |A \cap B| = |B - A| = 3$. Draw the accompanying Venn diagram.

1.24. Give examples of three sets A, B and C such that $B \neq C$ but $B - A = C - A$.

1.25. Give examples of three sets A, B and C such that

(a) $A \in B$, $A \subseteq C$ and $B \nsubseteq C$
(b) $B \in A$, $B \subset C$ and $A \cap C \neq \emptyset$
(c) $A \in B$, $B \subseteq C$ and $A \nsubseteq C$.

1.26. Let U be a universal set and let A and B be two subsets of U. Draw a Venn diagram for each of the following sets.

(a) $\overline{A \cup B}$ (b) $\overline{A} \cap \overline{B}$ (c) $\overline{A \cap B}$ (d) $\overline{A} \cup \overline{B}$.

What can you say about parts (a) and (b)? parts (c) and (d)?

1.27. Give an example of a universal set U, two sets A and B and accompanying Venn diagram such that $|A \cap B| = |A - B| = |B - A| = |\overline{A \cup B}| = 2$.

1.28. Let A, B and C be nonempty subsets of a universal set U. Draw a Venn diagram for each of the following set operations.

(a) $(C - B) \cup A$
(b) $C \cap (A - B)$.

1.29. Let $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.

(a) Determine which of the following are elements of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
(b) Determine $|A|$.
(c) Determine which of the following are subsets of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
For (d)–(i), determine the indicated sets.
(d) $\emptyset \cap A$
(e) $\{\emptyset\} \cap A$
(f) $\{\emptyset, \{\emptyset\}\} \cap A$
(g) $\emptyset \cup A$
(h) $\{\emptyset\} \cup A$
(i) $\{\emptyset, \{\emptyset\}\} \cup A$.

1.30. Let $A = \{x \in \mathbf{R} : |x - 1| \leq 2\}$, $B = \{x \in \mathbf{R} : |x| \geq 1\}$ and $C = \{x \in \mathbf{R} : |x + 2| \leq 3\}$.

(a) Express A, B and C using interval notation.
(b) Determine each of the following sets using interval notation:
$A \cup B$, $A \cap B$, $B \cap C$, $B - C$.

1.31. Give an example of four different sets A, B, C and D such that (1) $A \cup B = \{1, 2\}$ and $C \cap D = \{2, 3\}$ and (2) if B and C are interchanged and $\cup$ and $\cap$ are interchanged, then we get the same result.

1.32. Give an example of four different subsets A, B, C and D of $\{1, 2, 3, 4\}$ such that all intersections of two subsets are different.

1.33. Give an example of two nonempty sets A and B such that $\{A \cup B, A \cap B, A - B, B - A\}$ is the power set of some set.

1.34. Give an example of two subsets A and B of $\{1, 2, 3\}$ such that all of the following sets are different: $A \cup B$, $A \cup \overline{B}$, $\overline{A} \cup B$, $\overline{A} \cup \overline{B}$, $A \cap B$, $A \cap \overline{B}$, $\overline{A} \cap B$, $\overline{A} \cap \overline{B}$.

1.35. Give examples of a universal set U and sets A, B and C such that each of the following sets contains exactly one element: $A \cap B \cap C$, $(A \cap B) - C$, $(A \cap C) - B$, $(B \cap C) - A$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$, $\overline{A \cup B \cup C}$. Draw the accompanying Venn diagram.

Section 1.4: Indexed Collections of Sets

1.36. For a real number r, define S_r to be the interval $[r - 1, r + 2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$.

1.37. Let $A = \{1, 2, 5\}$, $B = \{0, 2, 4\}$, $C = \{2, 3, 4\}$ and $S = \{A, B, C\}$. Determine $\bigcup_{X \in S} X$ and $\bigcap_{X \in S} X$.

1.38. For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r - 1, r + 1]$ and C_r as the interval (r, ∞). For $S = \{1, 2, 4\}$, determine

(a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$
(b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$
(c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$.

1.39. Let $A = \{a, b, \ldots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let A_α consist of α and the two letters that follow it, where $A_y = \{y, z, a\}$ and $A_z = \{z, a, b\}$. Find a set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set S has the required properties.

1.40. For $i \in \mathbf{Z}$, let $A_i = \{i - 1, i + 1\}$. Determine the following:

(a) $\displaystyle\bigcup_{i=1}^{5} A_{2i}$ (b) $\displaystyle\bigcup_{i=1}^{5} (A_i \cap A_{i+1})$ (c) $\displaystyle\bigcup_{i=1}^{5} (A_{2i-1} \cap A_{2i+1})$.

1.41. For each of the following, find an indexed collection $\{A_n\}_{n \in \mathbf{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.

(a) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1]$
(b) $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ and $\bigcup_{n=1}^{\infty} A_n = \mathbf{Z}$.

1.42. For each of the following collections of sets, define a set A_n for each $n \in \mathbf{N}$ such that the indexed collection $\{A_n\}_{n \in \mathbf{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collection of sets.

(a) $\{[1, 2 + 1), [1, 2 + 1/2), [1, 2 + 1/3), \ldots\}$
(b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \ldots\}$.

1.43. For $r \in \mathbf{R}^+$, let $A_r = \{x \in \mathbf{R} : |x| < r\}$. Determine $\bigcup_{r \in \mathbf{R}^+} A_r$ and $\bigcap_{r \in \mathbf{R}^+} A_r$.

1.44. Each of the following sets is a subset of $A = \{1, 2, \ldots, 10\}$:
$A_1 = \{1, 5, 7, 9, 10\}$, $A_2 = \{1, 2, 3, 8, 9\}$, $A_3 = \{2, 4, 6, 8, 9\}$,
$A_4 = \{2, 4, 8\}$, $A_5 = \{3, 6, 7\}$, $A_6 = \{3, 8, 10\}$, $A_7 = \{4, 5, 7, 9\}$,
$A_8 = \{4, 5, 10\}$, $A_9 = \{4, 6, 8\}$, $A_{10} = \{5, 6, 10\}$,
$A_{11} = \{5, 8, 9\}$, $A_{12} = \{6, 7, 10\}$, $A_{13} = \{6, 8, 9\}$.
Find a set $I \subseteq \{1, 2, \ldots, 13\}$ such that for every two distinct elements $j, k \in I$, $A_j \cap A_k = \emptyset$ and $\left|\bigcup_{i \in I} A_i\right|$ is maximum.

1.45. For $n \in \mathbf{N}$, let $A_n = \left(-\frac{1}{n},\ 2 - \frac{1}{n},\right)$. Determine $\bigcup_{n \in \mathbf{N}} A_n$ and $\bigcap_{n \in \mathbf{N}} A_n$.

Section 1.5: Partitions of Sets

1.46. Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.
(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$ (b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$
(c) $S_3 = \{A\}$ (d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$
(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$.

1.47. Which of the following sets are partitions of $A = \{1, 2, 3, 4, 5\}$?
(a) $S_1 = \{\{1, 3\}, \{2, 5\}\}$ (b) $S_2 = \{\{1, 2\}, \{3, 4, 5\}\}$
(c) $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ (d) $S_4 = A$.

1.48. Let $A = \{1, 2, 3, 4, 5, 6\}$. Give an example of a partition S of A such that $|S| = 3$.

1.49. Give an example of a set A with $|A| = 4$ and two disjoint partitions S_1 and S_2 of A with $|S_1| = |S_2| = 3$.

1.50. Give an example of a partition of **N** into three subsets.

1.51. Give an example of a partition of **Q** into three subsets.

1.52. Give an example of three sets A, S_1 and S_2 such that S_1 is a partition of A, S_2 is a partition of S_1 and $|S_2| < |S_1| < |A|$.

1.53. Give an example of a partition of **Z** into four subsets.

1.54. Let $A = \{1, 2, \ldots, 12\}$. Give an example of a partition S of A satisfying the following requirements: (i) $|S| = 5$, (ii) there is a subset T of S such that $|T| = 4$ and $|\cup_{X \in T} X| = 10$ and (iii) there is no element $B \in S$ such that $|B| = 3$.

1.55. A set S is partitioned into two subsets S_1 and S_2. This produces a partition $\mathcal{P}_1$ of S where $\mathcal{P}_1 = \{S_1, S_2\}$ and so $|\mathcal{P}_1| = 2$. One of the sets in $\mathcal{P}_1$ is then partitioned into two subsets, producing a partition $\mathcal{P}_2$ of S with $|\mathcal{P}_2| = 3$. A total of $|\mathcal{P}_1|$ sets in $\mathcal{P}_2$ are partitioned into two subsets each, producing a partition $\mathcal{P}_3$ of S. Next, a total of $|\mathcal{P}_2|$ sets in $\mathcal{P}_3$ are partitioned into two subsets each, producing a partition $\mathcal{P}_4$ of S. This is continued until a partition $\mathcal{P}_6$ of S is produced. What is $|\mathcal{P}_6|$?

1.56. We mentioned that there are three ways that a collection $\mathcal{S}$ of subsets of a nonempty set A is defined to be a partition of A.
Definition 1 The collection $\mathcal{S}$ consists of pairwise disjoint nonempty subsets of A and every element of A belongs to a subset in $\mathcal{S}$.
Definition 2 The collection $\mathcal{S}$ consists of nonempty subsets of A and every element of A belongs to exactly one subset in $\mathcal{S}$.
Definition 3 The collection $\mathcal{S}$ consists of subsets of A satisfying the three properties (1) every subset in S is nonempty, (2) every two subsets of A are equal or disjoint and (3) the union of all subsets in S is A.

(a) Show that any collection $\mathcal{S}$ of subsets of A satisfying Definition 1 satisfies Definition 2.
(b) Show that any collection $\mathcal{S}$ of subsets of A satisfying Definition 2 satisfies Definition 3.
(c) Show that any collection $\mathcal{S}$ of subsets of A satisfying Definition 3 satisfies Definition 1.

Section 1.6: Cartesian Products of Sets

1.57. Let $A = \{x, y, z\}$ and $B = \{x, y\}$. Determine $A \times B$.

1.58. Let $A = \{1, \{1\}, \{\{1\}\}\}$. Determine $A \times A$.

1.59. For $A = \{a, b\}$, determine $A \times \mathcal{P}(A)$.

1.60. For $A = \{\emptyset, \{\emptyset\}\}$, determine $A \times \mathcal{P}(A)$.

1.61. For $A = \{1, 2\}$ and $B = \{\emptyset\}$, determine $A \times B$ and $\mathcal{P}(A) \times \mathcal{P}(B)$.

1.62. Describe the graph of the circle whose equation is $x^2 + y^2 = 4$ as a subset of $\mathbf{R} \times \mathbf{R}$.

1.63. List the elements of the set $S = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} : |x| + |y| = 3\}$. Plot the corresponding points in the Euclidean xy- plane.

1.64. For $A = \{1, 2\}$ and $B = \{1\}$, determine $\mathcal{P}(A \times B)$.

1.65. For $A = \{x \in \mathbf{R} : |x - 1| \le 2\}$ and $B = \{y \in \mathbf{R} : |y - 4| \le 2\}$, give a geometric description of the points in the xy-plane belonging to $A \times B$.

1.66. For $A = \{a \in \mathbf{R} : |a| \le 1\}$ and $B = \{b \in \mathbf{R} : |b| = 1\}$, give a geometric description of the points in the xy-plane belonging to $(A \times B) \cup (B \times A)$.

ADDITIONAL EXERCISES FOR CHAPTER 1

1.67. The set $T = \{2k + 1 : k \in \mathbf{Z}\}$ can be described as $T = \{\ldots, -3, -1, 1, 3, \ldots\}$. Describe the following sets in a similar manner.
(a) $A = \{4k + 3 : k \in \mathbf{Z}\}$
(b) $B = \{5k - 1 : k \in \mathbf{Z}\}$.

1.68. Let $S = \{-10, -9, \ldots, 9, 10\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x.
(a) $A = \{-10, -9, \ldots, -1, 1, \ldots, 9, 10\}$
(b) $B = \{-10, -9, \ldots, -1, 0\}$
(c) $C = \{-5, -4, \ldots, 0, 1, \ldots, 7\}$
(d) $D = \{-10, -9, \ldots, 4, 6, 7, \ldots, 10\}$.

1.69. Describe each of the following sets by listing its elements within braces.
(a) $\{x \in \mathbf{Z} : x^3 - 4x = 0\}$
(b) $\{x \in \mathbf{R} : |x| = -1\}$
(c) $\{m \in \mathbf{N} : 2 < m \le 5\}$
(d) $\{n \in \mathbf{N} : 0 \le n \le 3\}$
(e) $\{k \in \mathbf{Q} : k^2 - 4 = 0\}$
(f) $\{k \in \mathbf{Z} : 9k^2 - 3 = 0\}$
(g) $\{k \in \mathbf{Z} : 1 \le k^2 \le 10\}$.

1.70. Determine the cardinality of each of the following sets.
(a) $A = \{1, 2, 3, \{1, 2, 3\}, 4, \{4\}\}$
(b) $B = \{x \in \mathbf{R} : |x| = -1\}$
(c) $C = \{m \in \mathbf{N} : 2 < m \le 5\}$
(d) $D = \{n \in \mathbf{N} : n < 0\}$
(e) $E = \{k \in \mathbf{N} : 1 \le k^2 \le 100\}$
(f) $F = \{k \in \mathbf{Z} : 1 \le k^2 \le 100\}$.

1.71. For $A = \{-1, 0, 1\}$ and $B = \{x, y\}$, determine $A \times B$.

1.72. Let $U = \{1, 2, 3\}$ be the universal set and let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 3\}$. Determine the following.
(a) $(A \cup B) - (B \cap C)$
(b) $\overline{A}$
(c) $\overline{B \cup C}$
(d) $A \times B$.

1.73. Let $A = \{1, 2, \ldots, 10\}$. Give an example of two sets S and B such that $S \subseteq \mathcal{P}(A)$, $|S| = 4$, $B \in S$ and $|B| = 2$.

1.74. For $A = \{1\}$ and $C = \{1, 2\}$, give an example of a set B such that $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$.

1.75. Give examples of two sets A and B such that $A \cap \mathcal{P}(A) \in B$ and $\mathcal{P}(A) \subseteq A \cup B$.

1.76. Which of the following sets are equal?

$A = \{n \in \mathbf{Z} : -4 \le n \le 4\}$ $D = \{x \in \mathbf{Z} : x^3 = 4x\}$
$B = \{x \in \mathbf{N} : 2x + 2 = 0\}$ $E = \{-2, 0, 2\}$.
$C = \{x \in \mathbf{Z} : 3x - 2 = 0\}$

1.77. Let A and B be subsets of some unknown universal set U. Suppose that $\overline{A} = \{3, 8, 9\}$, $A - B = \{1, 2\}$, $B - A = \{8\}$ and $A \cap B = \{5, 7\}$. Determine U, A and B.

1.78. Let I denote the interval $[0, \infty)$. For each $r \in I$, define

$$A_r = \left\{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = r^2\right\}$$
$$B_r = \left\{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 \le r^2\right\}$$
$$C_r = \left\{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 > r^2\right\}.$$

(a) Determine $\bigcup_{r \in I} A_r$ and $\bigcap_{r \in I} A_r$.
(b) Determine $\bigcup_{r \in I} B_r$ and $\bigcap_{r \in I} B_r$.
(c) Determine $\bigcup_{r \in I} C_r$ and $\bigcap_{r \in I} C_r$.

1.79. Give an example of four sets A_1, A_2, A_3, A_4 such that $|A_i \cap A_j| = |i - j|$ for every two integers i and j with $1 \le i < j \le 4$.

1.80. (a) Give an example of two problems suggested by Exercise 1.79 (above).
(b) Solve one of the problems in (a).

1.81. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{1, 2, 3, 4, 5\}$. For the sets S and T described below, explain whether $|S| < |T|$, $|S| > |T|$ or $|S| = |T|$.

(a) Let B be the universal set and let S be the set of all subsets X of B for which $|X| \ne |\overline{X}|$. Let T be the set of 2-element subsets of C.
(b) Let S be the set of all partitions of the set A and let T be the set of 4-element subsets of C.
(c) Let $S = \{(b, a) : b \in B, a \in A, a + b \text{ is odd}\}$ and let T be the set of all nonempty proper subsets of A.

1.82. Give an example of a set $A = \{1, 2, \ldots, k\}$ for a smallest $k \in \mathbf{N}$ containing subsets A_1, A_2, A_3 such that $|A_i - A_j| = |A_j - A_i| = |i - j|$ for every two integers i and j with $1 \le i < j \le 3$.

1.83. (a) For $A = \{-3, -2, \ldots, 4\}$ and $B = \{1, 2, \ldots, 6\}$, determine $S = \{(a, b) \in A \times B : a^2 + b^2 = 25\}$.
(b) For $C = \{a \in B : (a, b) \in S\}$ and $D = \{b \in A : (a, b) \in S\}$, where A, B, S are the sets in (a), determine $C \times D$.

1.84. For $A = \{1, 2, 3\}$, let B be the set of 2-element sets belonging to $\mathcal{P}(A)$ and let C be the set consisting of the sets that are the intersections of two distinct elements of B. Determine $D = \mathcal{P}(C)$.

1.85. For a real number r, let $A_r = \{r, r + 1\}$. Let $S = \{x \in \mathbf{R} : x^2 + 2x - 1 = 0\}$.

(a) Determine $B = A_s \times A_t$ for the distinct elements $s, t \in S$, where $s < t$.
(b) Let $C = \{ab : (a, b) \in B\}$. Determine the sum of the elements of C.

2 Logic

In mathematics our goal is to seek the truth. Are there connections between two given mathematical concepts? If so, what are they? Under what conditions does an object possess a particular property? Finding answers to questions such as these is important, but we cannot be satisfied only with this. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here, however. It is the writer's responsibility to use the rules of logic to give a valid and clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved sufficiently well so that he or she will not only be able to understand a well-presented argument but can decide as well whether it is valid. Consequently, both writer and reader must be familiar with logic.

Although it is possible to spend a great deal of time studying logic, we will present only what we actually need and will instead use the majority of our time putting what we learn into practice.

2.1 Statements

In mathematics we are constantly dealing with statements. By a **statement** we mean a declarative sentence or assertion that is true or false (but not both). Statements therefore declare or assert the truth of something. Of course, the statements in which we will be primarily interested deal with mathematics. For example, the sentences

The integer 3 is odd.
The integer 57 is prime.

are statements (only the first of which is true).

Every statement has a **truth value**, namely **true** (denoted by T) or **false** (denoted by F). We often use P, Q and R to denote statements, or perhaps $P_1, P_2, \dots, P_n$ if there are several statements involved. We have seen that

$$P_1 : \text{The integer 3 is odd.}$$

and

$$P_2 : \text{The integer 57 is prime.}$$

are statements, where P_1 has truth value T and P_2 has truth value F.

Sentences that are imperative (commands) such as

Substitute the number 2 for x.
Find the derivative of $f(x) = e^{-x} \cos 2x$.

or are interrogative (questions) such as

Are these sets disjoint?
What is the derivative of $f(x) = e^{-x} \cos 2x$?

or are exclamatory such as

What an interesting question!
How difficult this problem is!

are not statements since these sentences are not declarative.

It may not be immediately clear whether a statement is true or false. For example, the sentence "The 100th digit in the decimal expansion of π is 7." is a statement, but it may be necessary to find this information in a Web site on the Internet to determine whether this statement is true. Indeed, for a sentence to be a statement, it is not a requirement that we be able to determine its truth value.

The sentence "The real number r is rational." is a statement *provided* we know what real number r is being referred to. Without this additional information, however, it is impossible to assign a truth value to it. This is an example of what is often referred to as an open sentence. In general, an **open sentence** is a declarative sentence that contains one or more variables, each variable representing a value in some prescribed set, called the **domain** of the variable, and which becomes a statement when values from their respective domains are substituted for these variables. For example, the open sentence "$3x = 12$" where the domain of x is the set of integers is a true statement only when $x = 4$.

An open sentence that contains a variable x is typically represented by $P(x)$, $Q(x)$ or $R(x)$. If $P(x)$ is an open sentence, where the domain of x is S, then we say $P(x)$ is an **open sentence over the domain** S. Also, $P(x)$ is a statement for each $x \in S$. For example, the open sentence

$$P(x): \ (x-3)^2 \leq 1$$

over the domain $\mathbf{Z}$ is a true statement when $x \in \{2, 3, 4\}$ and is a false statement otherwise.

Example 2.1 *For the open sentence*

$$P(x, y): \ |x+1| + |y| = 1$$

P
T
F

Q
T
F

P	Q
T	T
T	F
F	T
F	F

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Figure 2.1 Truth tables for one, two and three statements

in two variables, suppose that the domain of the variable x *is* $S = \{-2, -1, 0, 1\}$ *and the domain of the variable* y *is* $T = \{-1, 0, 1\}$. *Then*

$$P(-1, 1):\ |(-1) + 1| + |1| = 1$$

is a true statement, while

$$P(1, -1):\ |1 + 1| + |-1| = 1$$

is a false statement. In fact, $P(x, y)$ *is a true statement when*

$$(x, y) \in \{(-2, 0), (-1, -1), (-1, 1), (0, 0)\},$$

while $P(x, y)$ *is a false statement for all other elements* $(x, y) \in S \times T$. ♦

The possible truth values of a statement are often listed in a table, called a **truth table.** The truth tables for two statements P and Q are given in Figure 2.1. Since there are two possible truth values for each of P and Q, there are four possible combinations of truth values for P and Q. The truth table showing all these combinations is also given in Figure 2.1. If a third statement R is involved, then there are eight possible combinations of truth values for P, Q and R. This is displayed in Figure 2.1 as well. In general, a truth table involving n statements $P_1, P_2, \cdots, P_n$ contains 2^n possible combinations of truth values for these statements and a truth table showing these combinations would have n columns and 2^n rows. Much of the time, we will be dealing with two statements, usually denoted by P and Q; so the associated truth table will have four rows with the first two columns headed by P and Q. In this case, it is customary to consider the four combinations of the truth values in the order TT, TF, FT, FF, from top to bottom.

2.2 The Negation of a Statement

Much of the interest in integers and other familiar sets of numbers comes not only from the numbers themselves but from properties of the numbers that result by performing operations on them (such as taking their negatives, adding or multiplying them or combinations of these). Similarly, much of our interest in statements comes from investigating the truth or falseness of new statements that can be produced from one or more given statements by performing certain operations on them. Our first example concerns producing a new statement from a single given statement.

The **negation** of a statement P is the statement:

not P.

and is denoted by $\sim P$. Although $\sim P$ could always be expressed as

It is not the case that P**.**

there are usually better ways to express the statement $\sim P$.

Example 2.2 *For the statement*

P_1 : *The integer* 3 *is odd.*

described above, we have

$\sim P_1$: *It is not the case that the integer* 3 *is odd.*

but it would be much preferred to write

$\sim P_1$: *The integer* 3 *is not odd.*

or better yet to write

$\sim P_1$: *The integer* 3 *is even.*

Similarly, the negation of the statement

P_2 : *The integer* 57 *is prime.*

considered above is

$\sim P_2$: *The integer* 57 *is not prime.*

Note that $\sim P_1$ *is false, while* $\sim P_2$ *is true.* ♦

Indeed, the negation of a true statement is always false and the negation of a false statement is always true; that is, the truth value of $\sim P$ is opposite to that of P. This is summarized in Figure 2.2, which gives the truth table for $\sim P$ (in terms of the possible truth values of P).

P	$\sim P$
T	F
F	T

Figure 2.2 The truth table for negation

2.3 The Disjunction and Conjunction of Statements

For two given statements P and Q, a common way to produce a new statement from them is by inserting the word "or" or "and" between P and Q. The **disjunction** of the statements P and Q is the statement

P **or** Q

and is denoted by $P \vee Q$. The disjunction $P \vee Q$ is true if at least one of P and Q is true; otherwise, $P \vee Q$ is false. Therefore, $P \vee Q$ is true if exactly one of P and Q is true or if both P and Q are true.

Example 2.3 *For the statements*

P_1 : *The integer* 3 *is odd. and* P_2 : *The integer* 57 *is prime.*

described earlier, the disjunction is the new statement

$P_1 \vee P_2$: *Either* 3 *is odd or* 57 *is prime.*

which is true since at least one of P_1 *and* P_2 *is true (namely,* P_1 *is true). Of course, in this case exactly one of* P_1 *and* P_2 *is true.* ♦

For two statements P and Q, the truth table for $P \vee Q$ is shown in Figure 2.3. This truth table then describes precisely when $P \vee Q$ is true (or false).

Although the truth of "P or Q" allows for both P and Q to be true, there are instances when the use of "or" does not allow that possibility. For example, for an integer n, if we were to say "n is even or n is odd," then surely it is not possible for both "n is even" and "n is odd" to be true. When "or" is used in this manner, it is called the **exclusive or**. Suppose, for example, that $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, where $k \geq 2$, is a partition of a set S and x is some element of S. If

$$x \in S_1 \text{ or } x \in S_2$$

is true, then it is impossible for both $x \in S_1$ and $x \in S_2$ to be true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 2.3 The truth table for disjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 2.4 The truth table for conjunction

The **conjunction** of the statements P and Q is the statement:

P and Q

and is denoted by $P \wedge Q$. The conjunction $P \wedge Q$ is true only when both P and Q are true; otherwise, $P \wedge Q$ is false.

Example 2.4 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime., the statement*

$$P_1 \wedge P_2 : 3 \textit{ is odd and } 57 \textit{ is prime.}$$

is false since P_2 is false and so not both P_1 and P_2 are true. ♦

The truth table for the conjunction of two statements is shown in Figure 2.4.

2.4 The Implication

A statement formed from two given statements in which we will be most interested is the implication (also called the conditional). For statements P and Q, the **implication** (or **conditional**) is the statement

If P, then Q.

and is denoted by $P \Rightarrow Q$. In addition to the wording "If P, then Q," we also express $P \Rightarrow Q$ in words as

P implies Q.

The truth table for $P \Rightarrow Q$ is given in Figure 2.5.

Notice that $P \Rightarrow Q$ is false only when P is true and Q is false ($P \Rightarrow Q$ is true otherwise).

Example 2.5 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime., the implication*

$$P_1 \Rightarrow P_2 : \textit{If } 3 \textit{ is an odd integer, then } 57 \textit{ is prime.}$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 2.5 The truth table for implication

is a false statement. The implication

$$P_2 \Rightarrow P_1 : \textit{If } 57 \textit{ is prime, then } 3 \textit{ is odd.}$$

is true, however. ♦

While the truth tables for the negation $\sim P$, the disjunction $P \vee Q$ and the conjunction $P \wedge Q$ are probably what one would expect, this may not be so for the implication $P \Rightarrow Q$. There is ample justification, however, for the truth values in the truth table of $P \Rightarrow Q$. We illustrate this with an example.

Example 2.6 *A student is taking a math class (let's say this one) and is currently receiving a $B+$. He visits his instructor a few days before the final examination and asks her, "Is there any chance that I can get an A in this course?" His instructor looks through her grade book and says, "If you earn an A on the final exam, then you will receive an A for your final grade." We now check the truth or falseness of this implication based on the various combinations of truth values of the statements*

$$P : \textit{You earn an A on the final exam.}$$

and

$$Q : \textit{You receive an A for your final grade.}$$

which make up the implication.

Analysis Suppose first that P and Q are both true. That is, the student receives an A on his final exam and later learns that he got an A for his final grade in the course. Did his instructor tell the truth? I think we would all agree that she did. So if P and Q are both true, then so too is $P \Rightarrow Q$, which agrees with the first row of the truth table of Figure 2.5.

Second, suppose that P is true and Q is false. So the student got an A on his final exam but did not receive an A as a final grade, say he received a B. Certainly, his instructor did not do as she promised (as she will soon be reminded by her student). What she said was false, which agrees with the second row of the table in Figure 2.5.

Third, suppose that P is false and Q is true. In this case, the student did not get an A on his final exam (say he earned a B) *but* when he received his final grades, he learned (and was pleasantly surprised) that his final grade was an A. How could this happen? Perhaps his instructor was lenient. Perhaps the final exam was unusually difficult and a grade of B on it indicated an exceptionally good performance. Perhaps the instructor made a mistake. In any case, the instructor did not lie; so she told the truth. Indeed, she never promised anything if the student did not get an A on his final exam. This agrees with the third row of the table in Figure2.5.

Finally, suppose that P and Q are both false. That is, suppose the student did not get an A on his final exam and he also did not get an A for a final grade. The instructor did not lie here either. She only promised the student an A *if* he got an A on the final exam. Once again, she did not promise anything if the student did not get an A on the final exam. So the instructor told the truth and this agrees with the fourth and final row of the table. ♦

In summary then, the only situation for which $P \Rightarrow Q$ is false is when P is true and Q is false (so $\sim Q$ is true). That is, the truth tables for

$$\sim(P \Rightarrow Q) \text{ and } P \wedge (\sim Q)$$

are the same. We'll revisit this observation again soon.

We have already mentioned that the implication $P \Rightarrow Q$ can be expressed as both "If P, then Q" and "P implies Q." In fact, there are several ways of expressing $P \Rightarrow Q$ in words, namely:

If P, then Q.
Q if P.
P implies Q.
P only if Q.
P is sufficient for Q.
Q is necessary for P.

It is probably not surprising that the first three of these say the same thing, but perhaps not at all obvious that the last three say the same thing as the first three. Consider the statement "P only if Q." This says that P is true only under the condition that Q is true; in other words, it cannot be the case that P is true and Q is false. Thus it says that if P is true, then necessarily Q must be true. We can also see from this that the statement "Q is necessary for P" has the same meaning as "P only if Q." The statement "P is sufficient for Q" states that the truth of P is sufficient for the truth of Q. In other words, the truth of P implies the truth of Q; that is, "P implies Q."

2.5 More on Implications

We have just discussed four ways to create new statements from one or two given statements. In mathematics, however, we are often interested in declarative sentences containing variables and whose truth or falseness is only known once we have assigned values to the variables. The values assigned to the variables come from their respective domains. These sentences are, of course, precisely the sentences we have referred to as open sentences. Just as new statements can be formed from statements P and Q by negation, disjunction, conjunction or implication, new open sentences can be constructed from open sentences in the same manner.

Example 2.7 *Consider the open sentences*

$$P_1(x) : x = -3. \textit{ and } P_2(x) : |x| = 3,$$

where $x \in \mathbf{R}$, that is, where the domain of x is $\mathbf{R}$ in each case. We can then form the following open sentences:

$$\begin{aligned} \sim P_1(x) &: \ x \neq -3. \\ P_1(x) \vee P_2(x) &: \ x = -3 \textit{ or } |x| = 3. \\ P_1(x) \wedge P_2(x) &: \ x = -3 \textit{ and } |x| = 3. \\ P_1(x) \Rightarrow P_2(x) &: \ \textit{If } x = -3, \textit{ then } |x| = 3. \end{aligned}$$

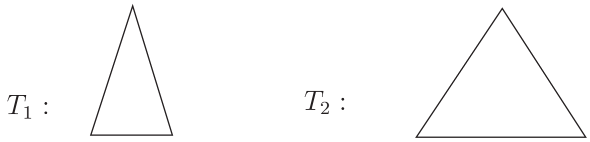

Figure 2.6 Isosceles and equilateral triangles

For a specific real number x, the truth value of each resulting statement can be determined. For example, $\sim P_1(-3)$ is a false statement, while each of the remaining sentences above results in a true statement when $x = -3$. Both $P_1(2) \vee P_2(2)$ and $P_1(2) \wedge P_2(2)$ are false statements. On the other hand, both $\sim P_1(2)$ and $P_1(2) \Rightarrow P_2(2)$ are true statements. In fact, for each real number $x \neq -3$, the implication $P_1(x) \Rightarrow P_2(x)$ is a true statement since $P_1(x) : x = -3$ is a false statement. Thus $P_1(x) \Rightarrow P_2(x)$ is true for all $x \in \mathbf{R}$. We will see that open sentences which result in true statements for all values of the domain will be especially interesting to us.

Listed below are various ways of wording the implication $P_1(x) \Rightarrow P_2(x)$:

If $x = -3$, then $|x| = 3$.
$|x| = 3$ if $x = -3$.
$x = -3$ implies that $|x| = 3$.
$x = -3$ only if $|x| = 3$.
$x = -3$ is sufficient for $|x| = 3$.
$|x| = 3$ is necessary for $x = -3$. ♦

We now consider another example, this time from geometry. You may recall that a triangle is called **equilateral** if the lengths of its three sides are the same, while a triangle is **isosceles** if the lengths of any two of its three sides are the same. Figure 2.6 shows an isosceles triangle T_1 and an equilateral triangle T_2. Actually, since the lengths of any two of the three sides of T_2 are the same, T_2 is isosceles as well. Indeed, this is precisely the fact we wish to discuss.

Example 2.8 *For a triangle T, let*

$$P(T) : T \text{ is equilateral. and } Q(T) : T \text{ is isosceles.}$$

Thus, $P(T)$ and $Q(T)$ are open sentences over the domain S of all triangles. Consider the implication $P(T) \Rightarrow Q(T)$, where the domain then of the variable T is the set S. For an equilateral triangle T_1, both $P(T_1)$ and $Q(T_1)$ are true statements and so $P(T_1) \Rightarrow Q(T_1)$ is a true statement as well. If T_2 is not an equilateral triangle, then $P(T_2)$ is a false statement and so $P(T_2) \Rightarrow Q(T_2)$ is true. Therefore, $P(T) \Rightarrow Q(T)$ is a true statement for all $T \in S$. We now state $P(T) \Rightarrow Q(T)$ in a variety of ways:

If T is an equilateral triangle, then T is isosceles.
A triangle T is isosceles if T is equilateral.
A triangle T being equilateral implies that T is isosceles.
A triangle T is equilateral only if T is isosceles.
For a triangle T to be isosceles, it is sufficient that T be equilateral.
For a triangle T to be equilateral, it is necessary that T be isosceles. ♦

Notice that at times we change the wording to make the sentence sound better. In general, the sentence P in the implication $P \Rightarrow Q$ is commonly referred to as the **hypothesis** or **premise** of $P \Rightarrow Q$, while Q is called the **conclusion** of $P \Rightarrow Q$.

It is often easier to deal with an implication when expressed in an "if, then" form. This allows us to identify the hypothesis and conclusion more easily. Indeed, since implications can be stated in a wide variety of ways (even in addition to those mentioned above), being able to reword an implication as "if, then" is especially useful. For example, the implication $P(T) \Rightarrow Q(T)$ described in Example 2.8 can be encountered in many ways, including the following:

- Let T be an equilateral triangle. Then T is isosceles.
- Suppose that T is an equilateral triangle. Then T is isosceles.
- Every equilateral triangle is isosceles.
- Whenever a triangle is equilateral, it is isosceles.

We now investigate the truth or falseness of implications involving open sentences for values of their variables.

Example 2.9 *Let $S = \{2, 3, 5\}$ and let*

$$P(n):\ n^2 - n + 1 \textit{ is prime. and } Q(n):\ n^3 - n + 1 \textit{ is prime.}$$

be open sentences over the domain S. Determine the truth or falseness of the implication $P(n) \Rightarrow Q(n)$ for each $n \in S$.

Solution In this case, we have the following:

$$\begin{array}{lll} P(2):\ 3 \text{ is prime.} & P(3):\ 7 \text{ is prime.} & P(5):\ 21 \text{ is prime.} \\ Q(2):\ 7 \text{ is prime.} & Q(3):\ 25 \text{ is prime.} & Q(5):\ 121 \text{ is prime.} \end{array}$$

Consequently, $P(2) \Rightarrow Q(2)$ and $P(5) \Rightarrow Q(5)$ are true, while $P(3) \Rightarrow Q(3)$ is false. ♦

Example 2.10 *Let $S = \{1, 2\}$ and let $T = \{-1, 4\}$. Also, let*

$$P(x, y):\ ||x + y| - |x - y|| = 2. \textit{ and } Q(x, y):\ x^{y+1} = y^x.$$

be open sentences, where the domain of the variable x is S and the domain of y is T. Determine the truth or falseness of the implication $P(x, y) \Rightarrow Q(x, y)$ for all $(x, y) \in S \times T$.

Solution For $(x, y) = (1, -1)$, we have

$$P(1, -1) \Rightarrow Q(1, -1):\ \text{If } 2 = 2, \text{ then } 1 = -1.$$

which is false. For $(x, y) = (1, 4)$, we have

$$P(1, 4) \Rightarrow Q(1, 4):\ \text{If } 2 = 2, \text{ then } 1 = 4.$$

which is also false. For $(x, y) = (2, -1)$, we have

$$P(2, -1) \Rightarrow Q(2, -1):\ \text{If } 2 = 2, \text{ then } 1 = 1.$$

which is true; while for $(x, y) = (2, 4)$, we have

$$P(2, 4) \Rightarrow Q(2, 4):\ \text{If } 2 = 4, \text{ then } 32 = 16.$$

which is true. ♦

2.6 The Biconditional

For statements (or open sentences) P and Q, the implication $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. The converse of an implication will often be of interest to us, either by itself or in conjunction with the original implication.

Example 2.11 *For the statements*

$$P_1 : \text{ 3 is an odd integer.} \qquad P_2 : \text{ 57 is prime.}$$

the converse of the implication

$$P_1 \Rightarrow P_2 : \text{ If 3 is an odd integer, then 57 is prime.}$$

is the implication

$$P_2 \Rightarrow P_1 : \text{ If 57 is prime, then 3 is an odd integer.}$$ ♦

For statements (or open sentences) P and Q, the conjunction

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

of the implication $P \Rightarrow Q$ and its converse is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. For statements P and Q, the truth table for $P \Leftrightarrow Q$ can therefore be determined. This is given in Figure 2.7. From this table, we see that $P \Leftrightarrow Q$ is true whenever the statements P and Q are both true or are both false, while $P \Leftrightarrow Q$ is false otherwise. That is, $P \Leftrightarrow Q$ is true precisely when P and Q have the same truth values.

The biconditional $P \Leftrightarrow Q$ is often stated as

P is equivalent to Q.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	***T***
T	F	F	T	***F***
F	T	T	F	***F***
F	F	T	T	***T***

P	$\star$	$Q\,P \Leftrightarrow Q$
T	T	***T***
T	F	***F***
F	T	***F***
F	F	***T***

Figure 2.7 The truth table for a biconditional

or

$$P \text{ if and only if } Q.$$

or as

$$P \text{ is a necessary and sufficient condition for } Q.$$

For statements P and Q, it then follows that the biconditional "P if and only if Q" is true only when P and Q have the same truth values.

Example 2.12 *The biconditional*

3 *is an odd integer if and only if* 57 *is prime.*

is false; while the biconditional

100 *is even if and only if* 101 *is prime.*

is true. Furthermore, the biconditional

5 *is even if and only if* 4 *is odd.*

is also true. ♦

The phrase "if and only if" occurs often in mathematics and we shall discuss this at greater length later. For the present, we consider two examples involving statements containing the phrase "if and only if."

Example 2.13 *We noted in Example 2.7 that for the open sentences*

$$P_1(x) : x = -3. \textit{ and } P_2(x) : |x| = 3.$$

over the domain $\mathbf{R}$*, the implication*

$$P_1(x) \Rightarrow P_2(x) : \textit{If } x = -3\textit{, then } |x| = 3.$$

is a true statement for each $x \in \mathbf{R}$*. However, the converse*

$$P_2(x) \Rightarrow P_1(x) : \textit{If } |x| = 3\textit{, then } x = -3.$$

is a false statement when $x = 3$ *since* $P_2(3)$ *is true and* $P_1(3)$ *is false. For all other real numbers* x*, the implication* $P_2(x) \Rightarrow P_1(x)$ *is true. Therefore, the biconditional*

$$P_1(x) \Leftrightarrow P_2(x) : \ x = -3 \textit{ if and only if } |x| = 3.$$

is false when $x = 3$ *and is true for all other real numbers* x*.* ♦

Example 2.14 *For the open sentences*

$$P(T) : \ T \textit{ is equilateral. and } Q(T) : \ T \textit{ is isosceles.}$$

over the domain S of all triangles, the converse of the implication

$$P(T) \Rightarrow Q(T): \textit{If } T \textit{ is equilateral, then } T \textit{ is isosceles.}$$

is the implication

$$Q(T) \Rightarrow P(T): \textit{If } T \textit{ is isosceles, then } T \textit{ is equilateral.}$$

We noted that $P(T) \Rightarrow Q(T)$ is a true statement for all triangles T, while $Q(T) \Rightarrow P(T)$ is a false statement when T is an isosceles triangle that is not equilateral. On the other hand, the second implication becomes a true statement for all other triangles T. Therefore, the biconditional

$$P(T) \Leftrightarrow Q(T): T \textit{ is equilateral if and only if } T \textit{ is isosceles.}$$

is false for all triangles that are isosceles and not equilateral, while it is true for all other triangles T. ♦

We now investigate the truth or falseness of biconditionals obtained by assigning to a variable each value in its domain.

Example 2.15 *Let $S = \{0, 1, 4\}$. Consider the following open sentences over the domain S:*

$$P(n): \frac{n(n+1)(2n+1)}{6} \textit{ is odd.}$$
$$Q(n): (n+1)^3 = n^3 + 1.$$

Determine three distinct elements a, b, c in S such that $P(a) \Rightarrow Q(a)$ is false, $Q(b) \Rightarrow P(b)$ is false, and $P(c) \Leftrightarrow Q(c)$ is true.

Solution Observe that

$$P(0): 0 \text{ is odd.} \quad P(1): 1 \text{ is odd.} \quad P(4): 30 \text{ is odd.}$$

$$Q(0): 1 = 1. \quad Q(1): 8 = 2. \quad Q(4): 125 = 65.$$

Thus $P(0)$ and $P(4)$ are false, while $P(1)$ is true. Also, $Q(1)$ and $Q(4)$ are false, while $Q(0)$ is true. Thus $P(1) \Rightarrow Q(1)$ and $Q(0) \Rightarrow P(0)$ are false, while $P(4) \Leftrightarrow Q(4)$ is true. Hence we may take $a = 1$, $b = 0$ and $c = 4$. ♦

Analysis Notice in Example 2.15 that both $P(0) \Leftrightarrow Q(0)$ and $P(1) \Leftrightarrow Q(1)$ are false biconditionals. Hence the value 4 in S is the only choice for c. ♦

2.7 Tautologies and Contradictions

The symbols $\sim$, $\vee$, $\wedge$, $\Rightarrow$ and $\Leftrightarrow$ are sometimes referred to as **logical connectives**. From given statements, we can use these logical connectives to form more intricate statements. For example, the statement $(P \vee Q) \wedge (P \vee R)$ is a statement formed from the given statements P, Q and R and the logical connectives $\vee$ and $\wedge$. We call $(P \vee Q) \wedge (P \vee R)$ a compound statement. More generally, a **compound statement** is a statement composed of one or more given statements (called **component statements** in this context) and at least one logical connective. For example, for a given component statement P, its negation $\sim P$ is a compound statement.

The compound statement $P \vee (\sim P)$, whose truth table is given in Figure 2.8, has the feature that it is true regardless of the truth value of P.

A compound statement S is called a **tautology** if it is true for all possible combinations of truth values of the component statements that comprise S. Hence $P \vee (\sim P)$ is a tautology, as is $(\sim Q) \vee (P \Rightarrow Q)$. This latter fact is verified in the truth table shown in Figure 2.9.

Letting

$$P_1 : \text{3 is odd. and } P_2 : \text{57 is prime.}$$

we see that not only is

57 is not prime, or 57 is prime if 3 is odd.

a true statement, but $(\sim P_2) \vee (P_1 \Rightarrow P_2)$ is true regardless of which statements P_1 and P_2 are being considered.

On the other hand, a compound statement S is called a **contradiction** if it is false for all possible combinations of truth values of the component statements that are used to form S. The statement $P \wedge (\sim P)$ is a contradiction, as is shown in Figure 2.10. Hence the statement

3 is odd and 3 is not odd.

is false.

Another example of a contradiction is $(P \wedge Q) \wedge (Q \Rightarrow (\sim P))$, which is verified in the truth table shown in Figure 2.11.

Indeed, if a compound statement S is a tautology, then its negation $\sim S$ is a contradiction.

P	$\sim P$	$P \vee (\sim P)$
T	F	$\boldsymbol{T}$
F	T	$\boldsymbol{T}$

Figure 2.8 An example of a tautology

P	Q	$\sim Q$	$P \Rightarrow Q$	$(\sim Q) \vee (P \Rightarrow Q)$
T	T	F	T	$\boldsymbol{T}$
T	F	T	F	$\boldsymbol{T}$
F	T	F	T	$\boldsymbol{T}$
F	F	T	T	$\boldsymbol{T}$

Figure 2.9 Another tautology

P	$\sim P$	$P \wedge (\sim P)$
T	F	$\boldsymbol{F}$
F	T	$\boldsymbol{F}$

Figure 2.10 An example of a contradiction

P	Q	$\sim P$	$P \wedge Q$	$Q \Rightarrow (\sim P)$	$(P \wedge Q) \wedge (Q \Rightarrow (\sim P))$
T	T	F	T	F	$\boldsymbol{F}$
T	F	F	F	T	$\boldsymbol{F}$
F	T	T	F	T	$\boldsymbol{F}$
F	F	T	F	T	$\boldsymbol{F}$

Figure 2.11 Another contradiction

2.8 Logical Equivalence

Figure 2.12 shows a truth table for the two statements $P \Rightarrow Q$ and $(\sim P) \vee Q$. The corresponding columns of these compound statements are identical; in other words, these two compound statements have exactly the same truth value for every combination of truth values of the statements P and Q. Let R and S be two compound statements involving the same component statements. Then R and S are called **logically equivalent** if R and S have the same truth values for all combinations of truth values of their component statements. If R and S are logically equivalent, then this is denoted by $R \equiv S$. Hence $P \Rightarrow Q$ and $(\sim P) \vee Q$ are logically equivalent and so $P \Rightarrow Q \equiv (\sim P) \vee Q$.

Another, even simpler, example of logical equivalence concerns $P \wedge Q$ and $Q \wedge P$. That $P \wedge Q \equiv Q \wedge P$ is verified in the truth table shown in Figure 2.13.

What is the practical significance of logical equivalence? Suppose that R and S are logically equivalent compound statements. Then we know that R and S have the same truth values for all possible combinations of truth values of their component statements. But this means that the biconditional $R \Leftrightarrow S$ is true for all possible combinations of truth values of their component statements and hence $R \Leftrightarrow S$ is a tautology. Conversely, if $R \Leftrightarrow S$ is a tautology, then R and S are logically equivalent.

Let R be a mathematical statement that we would like to show is true and suppose that R and some statement S are logically equivalent. If we can show that S is true, then R is true as well. For example, suppose that we want to verify the truth of an

P	Q	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	F	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F	$\boldsymbol{F}$	$\boldsymbol{F}$
F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
F	F	T	$\boldsymbol{T}$	$\boldsymbol{T}$

Figure 2.12 Verification of $P \Rightarrow Q \equiv (\sim P) \vee Q$

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	$\boldsymbol{F}$	$\boldsymbol{F}$
F	T	$\boldsymbol{F}$	$\boldsymbol{F}$
F	F	$\boldsymbol{F}$	$\boldsymbol{F}$

Figure 2.13 Verification of $P \wedge Q \equiv Q \wedge P$

implication $P \Rightarrow Q$. If we can establish the truth of the statement $(\sim P) \vee Q$, then the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ guarantees that $P \Rightarrow Q$ is true as well.

Example 2.16 *Returning to the mathematics instructor in Example 2.6 and whether she kept her promise that*

If you earn an A on the final exam, then you will receive an A for the final grade.

we need only know that the student did not receive an A on the final exam or the student received an A as a final grade to see that she kept her promise. ♦

Since the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$, verified in Figure 2.12, is especially important and we will have occasion to use this fact often, we state it as a theorem.

Theorem 2.17 *Let P and Q be two statements. Then*

$$P \Rightarrow Q \text{ and } (\sim P) \vee Q$$

are logically equivalent.

Let's return to the truth table in Figure 2.13, where we showed that $P \wedge Q$ and $Q \wedge P$ are logically equivalent for any two statements P and Q. In particular, this says that

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \text{ and } (Q \Rightarrow P) \wedge (P \Rightarrow Q)$$

are logically equivalent. Of course, $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is precisely what is called the biconditional of P and Q. Since $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ and $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ are logically equivalent, $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ represents the biconditional of P and Q as well. Since $Q \Rightarrow P$ can be written as "P if Q" and $P \Rightarrow Q$ can be expressed as "P only if Q," their conjunction can be written as "P if Q and P only if Q" or, more simply, as

$$P \text{ if and only if } Q.$$

Consequently, expressing $P \Leftrightarrow Q$ as "P if and only if Q" is justified. Furthermore, since $Q \Rightarrow P$ can be phrased as "P is necessary for Q" and $P \Rightarrow Q$ can be expressed as "P is sufficient for Q," writing $P \Leftrightarrow Q$ as "P is necessary and sufficient for Q" is likewise justified.

2.9 Some Fundamental Properties of Logical Equivalence

It probably comes as no surprise that the statements P and $\sim(\sim P)$ are logically equivalent. This fact is verified in Figure 2.14.

We mentioned in Figure 2.13 that, for two statements P and Q, the statements $P \wedge Q$ and $Q \wedge P$ are logically equivalent. There are other fundamental logical equivalences that we often encounter as well.

Theorem 2.18 *For statements P, Q and R,*

(1) *Commutative Laws*
  (*a*) $P \vee Q \equiv Q \vee P$
  (*b*) $P \wedge Q \equiv Q \wedge P$
(2) *Associative Laws*
  (*a*) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
  (*b*) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
(3) *Distributive Laws*
  (*a*) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  (*b*) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(4) *De Morgan's Laws*
  (*a*) $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
  (*b*) $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$.

Each part of Theorem 2.18 is verified by means of a truth table. We have already established the commutative law for conjunction (namely $P \wedge Q \equiv Q \wedge P$) in Figure 2.13. In Figure 2.15 $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ is verified by observing that the columns corresponding to the statements $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are identical.

The laws given in Theorem 2.18, together with other known logical equivalences, can be used to good advantage at times to prove other logical equivalences (without introducing a truth table).

Example 2.19 *Suppose we are asked to verify that*

$$\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$$

for every two statements P and Q. Using the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ from Theorem 2.17 and Theorem 2.18(4a), we see that

$$\sim (P \Rightarrow Q) \equiv \sim((\sim P) \vee Q) \equiv (\sim(\sim P)) \wedge (\sim Q) \equiv P \wedge (\sim Q), \qquad \textbf{(2.1)}$$

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

Figure 2.14 Verification of $P \equiv \sim (\sim P)$

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	*T*	*T*	*T*	*T*	*T*	***T***	***T***
T	*T*	*F*	*F*	*T*	*T*	***T***	***T***
T	*F*	*T*	*F*	*T*	*T*	***T***	***T***
T	*F*	*F*	*F*	*T*	*T*	***T***	***T***
F	*T*	*T*	*T*	*T*	*T*	***T***	***T***
F	*T*	*F*	*F*	*T*	*F*	***F***	***F***
F	*F*	*T*	*F*	*F*	*T*	***F***	***F***
F	*F*	*F*	*F*	*F*	*F*	***F***	***F***

Figure 2.15 Verification of the distributive law $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

implying that the statements $\sim (P \Rightarrow Q)$ *and* $P \wedge (\sim Q)$ *are logically equivalent, which we alluded to earlier.* ♦

It is important to keep in mind what we have said about logical equivalence. For example, the logical equivalence of $P \wedge Q$ and $Q \wedge P$ allows us to replace a statement of the type $P \wedge Q$ by $Q \wedge P$ without changing its truth value. As an additional example, according to De Morgan's Laws in Theorem 2.18, if it is not the case that an integer a is even or an integer b is even, then it follows that a and b are both odd.

Example 2.20 *Using the second of De Morgan's Laws and (2.1), we can establish a useful logically equivalent form of the negation of* $P \Leftrightarrow Q$ *by the following string of logical equivalences:*

$$\begin{aligned} \sim(P \Leftrightarrow Q) &\equiv \sim((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ &\equiv (\sim(P \Rightarrow Q)) \vee (\sim(Q \Rightarrow P)) \\ &\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)). \end{aligned}$$ ♦

What we have observed about the negation of an implication and a biconditional is repeated in the following theorem.

Theorem 2.21 *For statements* P *and* Q*,*

(*a*) $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$
(*b*) $\sim (P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$.

Example 2.22 *Once again, let's return to what the mathematics instructor in Example 2.6 said:*

If you earn an A on the final exam, then you will receive an A for your final grade.

If this instructor was not truthful, then it follows by Theorem 2.21(a) that

You earned an A on the final exam and did not receive A as your final grade.

Suppose, on the other hand, that the mathematics instructor had said:

> *If you earn an A on the final exam, then you will receive an A for the final grade—and that's the only way that you will get an A for a final grade.*

If this instructor was not truthful, then it follows by Theorem 2.21(b) that

> *Either you earned an A on the final exam and didn't receive A as your final grade or you received an A for your final grade and you didn't get an A on the final exam.* ♦

2.10 Quantified Statements

We have mentioned that if $P(x)$ is an open sentence over a domain S, then $P(x)$ is a statement for each $x \in S$. We illustrate this again.

Example 2.23 *If* $S = \{1, 2, \cdots, 7\}$, *then*

$$P(n): \ \frac{2n^2 + 5 + (-1)^n}{2} \ \textit{is prime.}$$

is a statement for each $n \in S$. *Therefore,*

$$\begin{aligned} P(1)&: \ 3 \textit{ is prime.} \\ P(2)&: \ 7 \textit{ is prime.} \\ P(3)&: \ 11 \textit{ is prime.} \\ P(4)&: \ 19 \textit{ is prime.} \end{aligned}$$

are true statements; while

$$\begin{aligned} P(5)&: \ 27 \textit{ is prime.} \\ P(6)&: \ 39 \textit{ is prime.} \\ P(7)&: \ 51 \textit{ is prime.} \end{aligned}$$

are false statements. ♦

There are other ways that an open sentence can be converted into a statement, namely by a method called **quantification**. Let $P(x)$ be an open sentence over a domain S. Adding the phrase "For every $x \in S$" to $P(x)$ produces a statement called a **quantified statement**. The phrase "for every" is referred to as the **universal quantifier** and is denoted by the symbol $\forall$. Other ways to express the universal quantifier are "for each" and "for all." This quantified statement is expressed in symbols by

$$\forall x \in S, P(x) \qquad \textbf{(2.2)}$$

and is expressed in words by

$$\text{For every } x \in S, P(x). \qquad \textbf{(2.3)}$$

The quantified statement (2.2) (or (2.3)) is true if $P(x)$ is true for every $x \in S$, while the quantified statement (2.2) is false if $P(x)$ is false for at least one element $x \in S$.

Another way to convert an open sentence $P(x)$ over a domain S into a statement through quantification is by the introduction of a quantifier called an existential quantifier. Each of the phrases *there exists*, *there is*, *for some* and *for at least one* is referred to as an **existential quantifier** and is denoted by the symbol ∃. The quantified statement

$$\exists x \in S,\ P(x) \tag{2.4}$$

can be expressed in words by

$$\text{There exists } x \in S \text{ such that } P(x). \tag{2.5}$$

The quantified statement (2.4) (or (2.5)) is true if $P(x)$ is true for at least one element $x \in S$, while the quantified statement (2.4) is false if $P(x)$ is false for all $x \in S$.

We now consider two quantified statements constructed from the open sentence we saw in Example 2.23.

Example 2.24 *For the open sentence*

$$P(n):\ \frac{2n^2+5+(-1)^n}{2} \textit{ is prime.}$$

over the domain $S = \{1, 2, \cdots, 7\}$*, the quantified statement*

$$\forall n \in S,\ P(n):\ \textit{For every } n \in S,\ \frac{2n^2+5+(-1)^n}{2} \textit{ is prime.}$$

is false since $P(5)$ *is false, for example; while the quantified statement*

$$\exists n \in S,\ P(n):\ \textit{There exists } n \in S \textit{ such that } \frac{2n^2+5+(-1)^n}{2} \textit{ is prime.}$$

is true since $P(1)$ *is true, for example.* ♦

The quantified statement $\forall x \in S,\ P(x)$ can also be expressed as

If $x \in S$, then $P(x)$.

Consider the open sentence $P(x):\ x^2 \geq 0$. over the set **R** of real numbers. Then

$$\forall x \in \mathbf{R},\ P(x)$$

or, equivalently,

$$\forall x \in \mathbf{R},\ x^2 \geq 0$$

can be expressed as

For every real number x, $x^2 \geq 0$.

or

If x is a real number, then $x^2 \geq 0$.

as well as

The square of every real number is nonnegative.

In general, the universal quantifier is used to claim that the statement resulting from a given open sentence is true when each value of the domain of the variable is assigned to the variable. Consequently, the statement $\forall x \in \mathbf{R}, x^2 \geq 0$ is true since $x^2 \geq 0$ is true for every real number x.

Suppose now that we were to consider the open sentence $Q(x) : x^2 \leq 0$. The statement $\forall x \in \mathbf{R}, Q(x)$ (that is, for every real number x, we have $x^2 \leq 0$) is false since, for example, $Q(1)$ is false. Of course, this means that its negation is true. If it were not the case that for every real number x, we have $x^2 \leq 0$, then there must exist some real number x such that $x^2 > 0$. This negation

There exists a real number x such that $x^2 > 0$.

can be written in symbols as

$$\exists x \in \mathbf{R}, x^2 > 0 \text{ or } \exists x \in \mathbf{R}, {\sim}Q(x).$$

More generally, if we are considering an open sentence $P(x)$ over a domain S, then

$$\sim (\forall x \in S, P(x)) \equiv \exists x \in S, {\sim}P(x).$$

Example 2.25 *Suppose that we are considering the set $A = \{1, 2, 3\}$ and its power set $\mathcal{P}(A)$, the set of all subsets of A. Then the quantified statement*

$$\textit{For every set } B \in \mathcal{P}(A),\ A - B \neq \emptyset. \qquad \textbf{(2.6)}$$

is false since for the subset $B = A = \{1, 2, 3\}$, we have $A - B = \emptyset$. The negation of the statement (2.6) is

$$\textit{There exists } B \in \mathcal{P}(A) \textit{ such that } A - B = \emptyset. \qquad \textbf{(2.7)}$$

The statement (2.7) is therefore true since for $B = A \in \mathcal{P}(A)$, we have $A - B = \emptyset$. The statement (2.6) can also be written as

$$\textit{If } B \subseteq A, \textit{ then } A - B \neq \emptyset. \qquad \textbf{(2.8)}$$

Consequently, the negation of (2.8) can be expressed as

There exists some subset B of A such that $A - B = \emptyset$. ♦

The existential quantifier is used to claim that at least one statement resulting from a given open sentence is true when the values of a variable are assigned from its domain. We know that for an open sentence $P(x)$ over a domain S, the quantified statement $\exists x \in S, P(x)$ is true provided $P(x)$ is a true statement for at least one element $x \in S$. Thus the statement $\exists x \in \mathbf{R}, x^2 > 0$ is true since, for example, $x^2 > 0$ is true for $x = 1$.

The quantified statement

$$\exists x \in \mathbf{R},\ 3x = 12$$

is therefore true since there is some real number x for which $3x = 12$, namely $x = 4$ has this property. (Indeed, $x = 4$ is the *only* real number for which $3x = 12$.) On the other hand, the quantified statement

$$\exists n \in \mathbf{Z},\ 4n - 1 = 0$$

is false as there is no integer n for which $4n - 1 = 0$. (Of course, $4n - 1 = 0$ when $n = 1/4$, but $1/4$ is not an integer.)

Suppose that $Q(x)$ is an open sentence over a domain S. If the statement $\exists x \in S$, $Q(x)$ is *not* true, then it must be the case that for every $x \in S$, $Q(x)$ is false. That is,

$$\sim (\exists x \in S,\ Q(x)) \equiv \forall x \in S, \sim Q(x)$$

is true. We illustrate this with a specific example.

Example 2.26 *The following statement contains the existential quantifier:*

$$\text{There exists a real number } x \text{ such that } x^2 = 3. \tag{2.9}$$

If we let $P(x) : x^2 = 3$, then (2.9) can be rewritten as $\exists x \in \mathbf{R}$, $P(x)$. The statement (2.9) is true since $P(x)$ is true when $x = \sqrt{3}$ (or when $x = -\sqrt{3}$). Hence the negation of (2.9) is:

$$\text{For every real number } x,\ x^2 \neq 3. \tag{2.10}$$

The statement (2.10) is therefore false. ♦

Let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. Then the quantified statement

For all $x \in S$ and $y \in T$, $P(x, y)$.

can be expressed symbolically as

$$\forall x \in S, \forall y \in T,\ P(x, y). \tag{2.11}$$

The negation of the statement (2.11) is

$$\begin{aligned} \sim (\forall x \in S, \forall y \in T,\ P(x, y)) &\equiv \exists x \in S,\ \sim (\forall y \in T,\ P(x, y)) \\ &\equiv \exists x \in S, \exists y \in T,\ \sim P(x, y). \end{aligned} \tag{2.12}$$

We now consider examples of quantified statements involving two variables.

Example 2.27 *Consider the statement*

$$\text{For every two real numbers } x \text{ and } y,\ x^2 + y^2 \geq 0. \tag{2.13}$$

If we let

$$P(x, y):\ x^2 + y^2 \geq 0$$

where the domain of both x and y is $\mathbf{R}$, then statement (2.13) can be expressed as

$$\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y) \tag{2.14}$$

or as

$$\forall x, y \in \mathbf{R},\ P(x, y).$$

Since $x^2 \geq 0$ and $y^2 \geq 0$ for all real numbers x and y, it follows that $x^2 + y^2 \geq 0$ and so $P(x, y)$ is true for all real numbers x and y. Thus the quantified statement (2.14) is true.

The negation of statement (2.14) is therefore

$$\sim (\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y)) \equiv \exists x \in \mathbf{R}, \exists y \in \mathbf{R},\ \sim P(x, y) \equiv \exists x, y \in \mathbf{R}, \sim P(x, y), \tag{2.15}$$

which, in words, is

$$\text{There exist real numbers } x \text{ and } y \text{ such that } x^2 + y^2 < 0. \tag{2.16}$$

The statement (2.16) is therefore false. ♦

For an open sentence containing two variables, the domains of the variables need not be the same.

Example 2.28 *Consider the statement*

$$\text{For every } s \in S \text{ and } t \in T,\ st + 2 \text{ is a prime.} \tag{2.17}$$

where the domain of the variable s is $S = \{1, 3, 5\}$ and the domain of the variable t is $T = \{3, 9\}$. If we let

$$Q(s, t):\ st + 2 \text{ is a prime.}$$

then the statement (2.17) can be expressed as

$$\forall s \in S, \forall t \in T,\ Q(s, t). \tag{2.18}$$

Since all of the statements

$Q(1, 3)$: $1 \cdot 3 + 2$ *is a prime.* $Q(3, 3)$: $3 \cdot 3 + 2$ *is a prime.*
$Q(5, 3)$: $5 \cdot 3 + 2$ *is a prime.*

$Q(1, 9)$: $1 \cdot 9 + 2$ *is a prime.* $Q(3, 9)$: $3 \cdot 9 + 2$ *is a prime.*
$Q(5, 9)$: $5 \cdot 9 + 2$ *is a prime.*

are true, the quantified statement (2.18) is true.

As we saw in (2.12), the negation of the quantified statement (2.18) is

$$\sim(\forall s \in S, \forall t \in T,\ Q(s, t)) \equiv \exists s \in S, \exists t \in T,\ \sim Q(s, t)$$

and so the negation of (2.17) is

$$\text{There exist } s \in S \text{ and } t \in T \text{ such that } st + 2 \text{ is not a prime.} \tag{2.19}$$

The statement (2.19) is therefore false. ♦

Again, let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. The quantified statement

$$\text{There exist } x \in S \text{ and } y \in T \text{ such that } P(x, y)$$

can be expressed in symbols as

$$\exists x \in S, \exists y \in T,\ P(x, y). \tag{2.20}$$

The negation of the statement (2.20) is then

$$\sim(\exists x \in S, \exists y \in T,\ P(x, y)) \equiv \forall x \in S,\ \sim(\exists y \in T,\ P(x, y))$$
$$\equiv \forall x \in S, \forall y \in T,\ \sim P(x, y). \qquad \textbf{(2.21)}$$

We now illustrate this situation.

Example 2.29 *Consider the open sentence*

$$R(s, t):\ |s - 1| + |t - 2| \le 2.$$

where the domain of the variable s is the set S of even integers and the domain of the variable t is the set T of odd integers. Then the quantified statement

$$\exists s \in S, \exists t \in T, R(s, t). \qquad \textbf{(2.22)}$$

can be expressed in words as

There exist an even integer s and an odd integer t such that $|s - 1| + |t - 2| \le 2$. **(2.23)**

Since $R(2, 3):\ 1 + 1 \le 2$ is true, the quantified statement (2.23) is true.

The negation of (2.22) is therefore

$$\sim(\exists s \in S, \exists t \in T,\ R(s, t)) \equiv \forall s \in S, \forall t \in T,\ \sim R(s, t). \qquad \textbf{(2.24)}$$

and so the negation of (2.22), in words, is

For every even integer s and every odd integer t, $|s - 1| + |t - 2| > 2$. **(2.25)**

The quantified statement (2.25) *is therefore false.* ◆

In the next two examples of negations of quantified statements, De Morgan's laws are also used.

Example 2.30 *The negation of*

For all integers a and b, if ab is even, then a is even and b is even.

is

There exist integers a and b such that ab is even and a or b is odd. ◆

Example 2.31 *The negation of*

There exists a rational number r such that $r \in A = \{\sqrt{2}, \pi\}$ or $r \in B = \{-\sqrt{2}, \sqrt{3}, e\}$.

is

For every rational number r, both $r \notin A$ and $r \notin B$. ◆

Quantified statements may contain both universal and existential quantifiers. While we present examples of these now, we will discuss these in more detail in Section 7.2.

Example 2.32 *Consider the open sentence*

$$P(a, b): \ ab = 1.$$

where the domain of both a and b is the set $\mathbf{Q}^+$ *of positive rational numbers. Then the quantified statement*

$$\forall a \in \mathbf{Q}^+, \exists b \in \mathbf{Q}^+, P(a, b) \tag{2.26}$$

can be expressed in words as

For every positive rational number a, there exists
a positive rational number b such that $ab = 1$.

It turns out that the quantified statement (2.26) is true. If we replace $\mathbf{Q}^+$ *by* $\mathbf{R}$*, then we have*

$$\forall a \in \mathbf{R}, \exists b \in \mathbf{R}, P(a, b). \tag{2.27}$$

The negation of this statement is

$$\begin{aligned} \sim (\forall a \in \mathbf{R}, \exists b \in \mathbf{R}, P(a, b)) &\equiv \exists a \in \mathbf{R}, \sim (\exists b \in \mathbf{R}, P(a, b)) \\ &\equiv \exists a \in \mathbf{R}, \forall b \in \mathbf{R}, \sim P(a, b), \end{aligned}$$

which, in words, says that

There exists a real number a such that for every real number b, $ab \neq 1$.

This negation is true since for $a = 0$ *and every real number b,* $ab = 0 \neq 1$*. Thus the quantified statement (2.27) is false.* ♦

Example 2.33 *Consider the open sentence*

$$Q(a, b): \ ab \text{ is odd.}$$

where the domain of both a and b is the set $\mathbf{N}$ *of positive integers. Then the quantified statement*

$$\exists a \in \mathbf{N}, \forall b \in \mathbf{N}, Q(a, b), \tag{2.28}$$

expressed in words, is

There exists a positive integer a such that for every positive integer b, ab is odd.

The statement (2.28) turns out to be false. The negation of (2.28), in symbols, is

$$\begin{aligned} \sim (\exists a \in \mathbf{N}, \forall b \in \mathbf{N}, Q(a, b)) &\equiv \forall a \in \mathbf{N}, \sim (\forall b \in \mathbf{N}, Q(a, b)) \\ &\equiv \forall a \in \mathbf{N}, \exists b \in \mathbf{N}, \sim Q(a, b). \end{aligned}$$

In words, this says

For every positive integer a, there exists a positive integer b such that ab is even.

This statement, therefore, is true. ♦

Suppose that $P(x, y)$ is an open sentence, where the domain of x is S and the domain of y is T. Then the quantified statement

$$\forall x \in S, \exists y \in T, P(x, y)$$

is true if $\exists y \in T, P(x, y)$ is true for each $x \in S$. This means that for every $x \in S$, there is some $y \in T$ for which $P(x, y)$ is true.

Example 2.34 *Consider the open sentence*

$$P(x, y)\text{: } x + y \textit{ is prime.}$$

where the domain of x is $S = \{2, 3\}$ and the domain of y is $T = \{3, 4\}$. The quantified statement

$$\forall x \in S, \exists y \in T, P(x, y),$$

expressed in words, is

For every $x \in S$, there exists $y \in T$ such that $x + y$ is prime.

This statement is true. For $x = 2$, $P(2, 3)$ is true and for $x = 3$, $P(3, 4)$ is true. ♦

Suppose that $Q(x, y)$ is an open sentence, where S is the domain of x and T is the domain of y. The quantified statement

$$\exists x \in S, \forall y \in T, Q(x, y)$$

is true if $\forall y \in T, Q(x, y)$ is true for some $x \in S$. This means that for some element x in S, the open sentence $Q(x, y)$ is true for all $y \in T$.

Example 2.35 *Consider the open sentence*

$$Q(x, y)\text{: } x + y \textit{ is prime.}$$

where the domain of x is $S = \{3, 5, 7\}$ and the domain of y is $T = \{2, 6, 8, 12\}$. The quantified statement

$$\exists x \in S, \forall y \in T, Q(x, y), \tag{2.29}$$

expressed in words, is

There exists some $x \in S$ such that for every $y \in T$, $x + y$ is prime.

For $x = 5$, all of the numbers $5 + 2$, $5 + 6$, $5 + 8$, and $5 + 12$ are prime. Consequently, the quantified statement (2.29) is true. ♦

Let's review symbols that we have introduced in this chapter:

$\sim$	negation (not)
$\vee$	disjunction (or)
$\wedge$	conjunction (and)
$\Rightarrow$	implication
$\Leftrightarrow$	biconditional
$\forall$	universal quantifier (for every)
$\exists$	existential quantifier (there exists)

2.11 Characterizations of Statements

Let's return to the biconditional $P \Leftrightarrow Q$. Recall that $P \Leftrightarrow Q$ represents the compound statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Earlier, we described how this compound statement can be expressed as

$$P \text{ if and only if } Q.$$

Many mathematicians abbreviate the phrase "if and only if" by writing "iff." Although "iff" is informal and, of course, is not a word, its use is common and you should be familiar with it.

Recall that whenever you see

$$P \text{ if and only if } Q.$$

or

$$P \text{ is necessary and sufficient for } Q.$$

this means

$$\text{If } P \text{ then } Q \textit{ and } \text{ if } Q \text{ then } P.$$

Example 2.36 *Suppose that*

$$P(x): x = -3. \textit{ and } Q(x): |x| = 3.$$

where $x \in \mathbf{R}$. Then the biconditional $P(x) \Leftrightarrow Q(x)$ can be expressed as

$$x = -3 \textit{ if and only if } |x| = 3.$$

or

$$x = -3 \textit{ is necessary and sufficient for } |x| = 3.$$

or, perhaps better, as

$$x = -3 \textit{ is a necessary and sufficient condition for } |x| = 3.$$

Let's now consider the quantified statement $\forall x \in \mathbf{R},\ P(x) \Leftrightarrow Q(x)$. This statement is false because $P(3) \Leftrightarrow Q(3)$ is false. ♦

Suppose that some concept (or object) is expressed in an open sentence $P(x)$ over a domain S and $Q(x)$ is another open sentence over the domain S concerning this concept. We say that this concept is **characterized** by $Q(x)$ if $\forall x \in S, P(x) \Leftrightarrow Q(x)$ is a true statement. The statement $\forall x \in S, P(x) \Leftrightarrow Q(x)$ is then called a **characterization** of this concept. For example, *irrational numbers* are defined as real numbers that are not rational and are characterized as real numbers whose decimal expansions are nonrepeating. This provides a characterization of irrational numbers:

A real number r is irrational if and only if r has a nonrepeating decimal expansion.

We saw that equilateral triangles are defined as triangles whose sides are equal. They are characterized however as triangles whose angles are equal. Therefore, we have the characterization:

A triangle T is equilateral if and only if T has three equal angles.

You might think that equilateral triangles are also characterized as those triangles having three equal sides but the associated biconditional:

A triangle T is equilateral if and only if T has three equal sides.

is not a characterization of equilateral triangles. Indeed, this is the definition we gave of equilateral triangles. A characterization of a concept then gives an alternative, but equivalent, way of looking at this concept. Characterizations are often valuable in studying concepts or in proving other results. We will see examples of this in future chapters.

We mentioned that the following biconditional, though true, is not a characterization: A triangle T is equilateral if and only if T has three equal sides. Although this is the definition of equilateral triangles, mathematicians rarely use the phrase "if and only if" in a definition since this is what is meant in a definition. That is, a triangle is defined to be equilateral if it has three equal sides. Consequently, a triangle with three equal sides is equilateral but a triangle that does not have three equal sides is not equilateral.

EXERCISES FOR CHAPTER 2

Section 2.1: Statements

2.1. Which of the following sentences are statements? For those that are, indicate the truth value.

(a) The integer 123 is prime.
(b) The integer 0 is even.
(c) Is $5 \times 2 = 10$?
(d) $x^2 - 4 = 0$.
(e) Multiply $5x + 2$ by 3.
(f) $5x + 3$ is an odd integer.
(g) What an impossible question!

2.2. Consider the sets A, B, C and D below. Which of the following statements are true? Give an explanation for each false statement.

$$A = \{1, 4, 7, 10, 13, 16, \ldots\} \quad C = \{x \in \mathbf{Z} : x \text{ is prime and } x \neq 2\}$$
$$B = \{x \in \mathbf{Z} : x \text{ is odd}\} \quad D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$$

(a) $25 \in A$ (b) $33 \in D$ (c) $22 \notin A \cup D$ (d) $C \subseteq B$ (e) $\emptyset \in B \cap D$ (f) $53 \notin C$.

2.3. Which of the following statements are true? Give an explanation for each false statement.
(a) $\emptyset \in \emptyset$ (b) $\emptyset \in \{\emptyset\}$ (c) $\{1, 3\} = \{3, 1\}$
(d) $\emptyset = \{\emptyset\}$ (e) $\emptyset \subset \{\emptyset\}$ (f) $1 \subseteq \{1\}$.

2.4. Consider the open sentence $P(x) : x(x - 1) = 6$ over the domain $\mathbf{R}$.

(a) For what values of x is $P(x)$ a true statement?
(b) For what values of x is $P(x)$ a false statement?

2.5. For the open sentence $P(x) : 3x - 2 > 4$ over the domain $\mathbf{Z}$, determine:

(a) the values of x for which $P(x)$ is true.
(b) the values of x for which $P(x)$ is false.

2.6. For the open sentence $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}(\{1, 2, 4\})$, determine:

(a) all $A \in S$ for which $P(A)$ is true.
(b) all $A \in S$ for which $P(A)$ is false.
(c) all $A \in S$ for which $A \cap \{1, 2, 3\} = \emptyset$.

2.7. Let $P(n)$: n and $n + 2$ are primes. be an open sentence over the domain $\mathbf{N}$. Find six positive integers n for which $P(n)$ is true. If $n \in \mathbf{N}$ such that $P(n)$ is true, then the two integers $n, n + 2$ are called **twin primes**. It has been conjectured that there are infinitely many twin primes.

2.8. Let $P(n) : \frac{n^2+5n+6}{2}$ is even.

(a) Find a set S_1 of three integers such that $P(n)$ is an open sentence over the domain S_1 and $P(n)$ is true for each $n \in S_1$.
(b) Find a set S_2 of three integers such that $P(n)$ is an open sentence over the domain S_2 and $P(n)$ is false for each $n \in S_2$.

2.9. Find an open sentence $P(n)$ over the domain $S = \{3, 5, 7, 9\}$ such that $P(n)$ is true for half of the integers in S and false for the other half.

2.10. Find two open sentences $P(n)$ and $Q(n)$, both over the domain $S = \{2, 4, 6, 8\}$, such that $P(2)$ and $Q(2)$ are both true, $P(4)$ and $Q(4)$ are both false, $P(6)$ is true and $Q(6)$ is false, while $P(8)$ is false and $Q(8)$ is true.

Section 2.2: The Negation of a Statement

2.11. State the negation of each of the following statements.

(a) $\sqrt{2}$ is a rational number.
(b) 0 is not a negative integer.
(c) 111 is a prime number.

2.12. Complete the truth table in Figure 2.16.

2.13. State the negation of each of the following statements.

(a) The real number r is at most $\sqrt{2}$.
(b) The absolute value of the real number a is less than 3.
(c) Two angles of the triangle are 45°.

P	Q	$\sim P$	$\sim Q$
T	T		
T	F		
F	T		
F	F		

Figure 2.16 The truth table for Exercise 2.12.

(d) The area of the circle is at least 9π.
(e) Two sides of the triangle have the same length.
(f) The point P in the plane lies outside of the circle C.

2.14. State the negation of each of the following statements.

(a) At least two of my library books are overdue.
(b) One of my two friends misplaced his homework assignment.
(c) No one expected that to happen.
(d) It's not often that my instructor teaches that course.
(e) It's surprising that two students received the same exam score.

Section 2.3: The Disjunction and Conjunction of Statements

2.15. Complete the truth table in Figure 2.17.

P	Q	$\sim Q$	$P \wedge (\sim Q)$
T	T		
T	F		
F	T		
F	F		

Figure 2.17 The truth table for Exercise 2.15

2.16. For the sets $A = \{1, 2, \cdots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, consider the statements

$$P\colon\ A \subseteq B. \quad Q\colon\ |A - B| = 6.$$

Determine which of the following statements are true.
(a) $P \vee Q$ (b) $P \vee (\sim Q)$ (c) $P \wedge Q$
(d) $(\sim P) \wedge Q$ (e) $(\sim P) \vee (\sim Q)$.

2.17. Let P: 15 is odd. and Q : 21 is prime. State each of the following in words, and determine whether they are true or false.
(a) $P \vee Q$ (b) $P \wedge Q$ (c) $(\sim P) \vee Q$ (d) $P \wedge (\sim Q)$.

2.18. Let $S = \{1, 2, \ldots, 6\}$ and let

$$P(A) : A \cap \{2, 4, 6\} = \emptyset. \text{ and } Q(A) : A \neq \emptyset.$$

be open sentences over the domain $\mathcal{P}(S)$.

(a) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \wedge Q(A)$ is true.

(b) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \vee (\sim Q(A))$ is true.
(c) Determine all $A \in \mathcal{P}(S)$ for which $(\sim P(A)) \wedge (\sim Q(A))$ is true.

Section 2.4: The Implication

2.19. Consider the statements P: 17 is even. and Q: 19 is prime. Write each of the following statements in words and indicate whether it is true or false.
(a) $\sim P$ (b) $P \vee Q$ (c) $P \wedge Q$ (d) $P \Rightarrow Q$.

2.20. For statements P and Q, construct a truth table for $(P \Rightarrow Q) \Rightarrow (\sim P)$.

2.21. Consider the statements P : $\sqrt{2}$ is rational. and Q : $22/7$ is rational. Write each of the following statements in words and indicate whether it is true or false.
(a) $P \Rightarrow Q$ (b) $Q \Rightarrow P$ (c) $(\sim P) \Rightarrow (\sim Q)$ (d) $(\sim Q) \Rightarrow (\sim P)$.

2.22. Consider the statements:

P: $\sqrt{2}$ is rational. Q: $\frac{2}{3}$ is rational. R: $\sqrt{3}$ is rational.

Write each of the following statements in words and indicate whether the statement is true or false.
(a) $(P \wedge Q) \Rightarrow R$ (b) $(P \wedge Q) \Rightarrow (\sim R)$
(c) $((\sim P) \wedge Q) \Rightarrow R$ (d) $(P \vee Q) \Rightarrow (\sim R)$.

2.23. Suppose that $\{S_1, S_2\}$ is a partition of a set S and $x \in S$. Which of the following are true?

(a) If we know that $x \notin S_1$, then x must belong to S_2.
(b) It's possible that $x \notin S_1$ and $x \notin S_2$.
(c) Either $x \notin S_1$ or $x \notin S_2$.
(d) Either $x \in S_1$ or $x \in S_2$.
(e) It's possible that $x \in S_1$ and $x \in S_2$.

2.24. Two sets A and B are nonempty disjoint subsets of a set S. If $x \in S$, then which of the following are true?

(a) It's possible that $x \in A \cap B$.
(b) If x is an element of A, then x can't be an element of B.
(c) If x is not an element of A, then x must be an element of B.
(d) It's possible that $x \notin A$ and $x \notin B$.
(e) For each nonempty set C, either $x \in A \cap C$ or $x \in B \cap C$.
(f) For some nonempty set C, both $x \in A \cup C$ and $x \in B \cup C$.

2.25. A college student makes the following statement:

> If I receive an A in both Calculus I and Discrete Mathematics this semester, then I'll take either Calculus II or Computer Programming this summer.

For each of the following, determine whether this statement is true or false.

(a) The student doesn't get an A in Calculus I but decides to take Calculus II this summer anyway.
(b) The student gets an A in both Calculus I and Discrete Mathematics but decides not to take any class this summer.
(c) The student does not get an A in Calculus I and decides not to take Calculus II but takes Computer Programming this summer.
(d) The student gets an A in both Calculus I and Discrete Mathematics and decides to take both Calculus II and Computer Programming this summer.
(e) The student gets an A in neither Calculus I nor Discrete Mathematics and takes neither Calculus II nor Computer Programming this summer.

2.26. A college student makes the following statement:

If I don't see my advisor today, then I'll see her tomorrow.

For each of the following, determine whether this statement is true or false.

(a) The student doesn't see his advisor either day.
(b) The student sees his advisor both days.
(c) The student sees his advisor on one of the two days.
(d) The student doesn't see his advisor today and waits until next week to see her.

2.27. The instructor of a computer science class announces to her class that there will be a well-known speaker on campus later that day. Four students in the class are Alice, Ben, Cindy and Don. Ben says that he'll attend the lecture if Alice does. Cindy says that she'll attend the talk if Ben does. Don says that he will go to the lecture if Cindy does. That afternoon exactly two of the four students attend the talk. Which two students went to the lecture?

2.28. Consider the statement (implication):

If Bill takes Sam to the concert, then Sam will take Bill to dinner.

Which of the following implies that this statement is true?

(a) Sam takes Bill to dinner only if Bill takes Sam to the concert.
(b) Either Bill doesn't take Sam to the concert or Sam takes Bill to dinner.
(c) Bill takes Sam to the concert.
(d) Bill takes Sam to the concert and Sam takes Bill to dinner.
(e) Bill takes Sam to the concert and Sam doesn't take Bill to dinner.
(f) The concert is canceled.
(g) Sam doesn't attend the concert.

2.29. Let P and Q be statements. Which of the following implies that $P \vee Q$ is false?
(a) $(\sim P) \vee (\sim Q)$ is false. (b) $(\sim P) \vee Q$ is true.
(c) $(\sim P) \wedge (\sim Q)$ is true. (d) $Q \Rightarrow P$ is true. (e) $P \wedge Q$ is false.

Section 2.5: More on Implications

2.30. Consider the open sentences $P(n)$: $5n + 3$ is prime. and $Q(n)$: $7n + 1$ is prime., both over the domain $\mathbf{N}$.

(a) State $P(n) \Rightarrow Q(n)$ in words.
(b) State $P(2) \Rightarrow Q(2)$ in words. Is this statement true or false?
(c) State $P(6) \Rightarrow Q(6)$ in words. Is this statement true or false?

2.31. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine the truth value of $P(x) \Rightarrow Q(x)$ for each $x \in S$.

(a) $P(x) : |x| = 4;\ Q(x) : x = 4;\ S = \{-4, -3, 1, 4, 5\}$.
(b) $P(x) : x^2 = 16;\ Q(x) : |x| = 4;\ S = \{-6, -4, 0, 3, 4, 8\}$.
(c) $P(x) : x > 3;\ Q(x) : 4x - 1 > 12;\ S = \{0, 2, 3, 4, 6\}$.

2.32. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.

(a) $P(x) : x - 3 = 4;\ Q(x) : x \geq 8;\ S = \mathbf{R}$.
(b) $P(x) : x^2 \geq 1;\ Q(x) : x \geq 1;\ S = \mathbf{R}$.
(c) $P(x) : x^2 \geq 1;\ Q(x) : x \geq 1;\ S = \mathbf{N}$.
(d) $P(x) : x \in [-1, 2];\ Q(x) : x^2 \leq 2;\ S = [-1, 1]$.

2.33. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbf{Z}$. Determine the truth value of $P(x, y) \Rightarrow Q(x, y)$ for the given values of x and y.

(a) $P(x, y)$: $x^2 - y^2 = 0$. and $Q(x, y)$: $x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.

(b) $P(x, y)$: $|x| = |y|$. and $Q(x, y)$: $x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.

(c) $P(x, y)$: $x^2 + y^2 = 1$. and $Q(x, y)$: $x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$.

2.34. Each of the following describes an implication. Write the implication in the form "if, then."

(a) Any point on the straight line with equation $2y + x - 3 = 0$ whose x-coordinate is an integer also has an integer for its y-coordinate.
(b) The square of every odd integer is odd.
(c) Let $n \in \mathbf{Z}$. Whenever $3n + 7$ is even, n is odd.
(d) The derivative of the function $f(x) = \cos x$ is $f'(x) = -\sin x$.
(e) Let C be a circle of circumference 4π. Then the area of C is also 4π.
(f) The integer n^3 is even only if n is even.

Section 2.6: The Biconditional

2.35. Let P : 18 is odd. and Q : 25 is even. State $P \Leftrightarrow Q$ in words. Is $P \Leftrightarrow Q$ true or false?

2.36. Let $P(x)$: x is odd. and $Q(x)$: x^2 is odd. be open sentences over the domain $\mathbf{Z}$. State $P(x) \Leftrightarrow Q(x)$ in two ways: (1) using "if and only if" and (2) using "necessary and sufficient."

2.37. For the open sentences $P(x)$: $|x - 3| < 1$. and $Q(x)$: $x \in (2, 4)$. over the domain $\mathbf{R}$, state the biconditional $P(x) \Leftrightarrow Q(x)$ in two different ways.

2.38. Consider the open sentences:

$$P(x) : x = -2. \text{ and } Q(x) : x^2 = 4.$$

over the domain $S = \{-2, 0, 2\}$. State each of the following in words and determine all values of $x \in S$ for which the resulting statements are true.

(a) $\sim P(x)$ (b) $P(x) \vee Q(x)$ (c) $P(x) \wedge Q(x)$ (d) $P(x) \Rightarrow Q(x)$
(e) $Q(x) \Rightarrow P(x)$ (f) $P(x) \Leftrightarrow Q(x)$.

2.39. For the following open sentences $P(x)$ and $Q(x)$ over a domain S, determine all values of $x \in S$ for which the biconditional $P(x) \Leftrightarrow Q(x)$ is true.

(a) $P(x) : |x| = 4$; $Q(x) : x = 4$; $S = \{-4, -3, 1, 4, 5\}$.
(b) $P(x) : x \geq 3$; $Q(x) : 4x - 1 > 12$; $S = \{0, 2, 3, 4, 6\}$.
(c) $P(x) : x^2 = 16$; $Q(x) : x^2 - 4x = 0$; $S = \{-6, -4, 0, 3, 4, 8\}$.

2.40. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbf{Z}$. Determine the truth value of $P(x, y) \Leftrightarrow Q(x, y)$ for the given values of x and y.

(a) $P(x, y) : x^2 - y^2 = 0$ and; $Q(x, y) : x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.

(b) $P(x, y) : |x| = |y|$ and; $Q(x, y) : x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.

(c) $P(x, y) : x^2 + y^2 = 1$ and; $Q(x, y) : x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$.

2.41. Determine all values of n in the domain $S = \{1, 2, 3\}$ for which the following is a true statement:
A necessary and sufficient condition for $\frac{n^3+n}{2}$ to be even is that $\frac{n^2+n}{2}$ is odd.

2.42. Determine all values of n in the domain $S = \{2, 3, 4\}$ for which the following is a true statement: The integer $\frac{n(n-1)}{2}$ is odd if and only if $\frac{n(n+1)}{2}$ is even.

2.43. Let $S = \{1, 2, 3\}$. Consider the following open sentences over the domain S:

$$P(n): \ \frac{(n+4)(n+5)}{2} \text{ is odd.}$$
$$Q(n): \ 2^{n-2} + 3^{n-2} + 6^{n-2} > (2.5)^{n-1}.$$

Determine three distinct elements a, b, c in S such that $P(a) \Rightarrow Q(a)$ is false, $Q(b) \Rightarrow P(b)$ is false, and $P(c) \Leftrightarrow Q(c)$ is true.

2.44. Let $S = \{1, 2, 3, 4\}$. Consider the following open sentences over the domain S:

$$P(n): \ \frac{n(n-1)}{2} \text{ is even.}$$
$$Q(n): \ 2^{n-2} - (-2)^{n-2} \text{ is even.}$$
$$R(n): \ 5^{n-1} + 2^n \text{ is prime.}$$

Determine four distinct elements a, b, c, d in S such that
(i) $P(a) \Rightarrow Q(a)$ is false; (ii) $Q(b) \Rightarrow P(b)$ is true;
(iii) $P(c) \Leftrightarrow R(c)$ is true; (iv) $Q(d) \Leftrightarrow R(d)$ is false.

2.45. Let $P(n)$: $2^n - 1$ is a prime. and $Q(n)$: n is a prime. be open sentences over the domain $S = \{2, 3, 4, 5, 6, 11\}$. Determine all values of $n \in S$ for which $P(n) \Leftrightarrow Q(n)$ is a true statement.

Section 2.7: Tautologies and Contradictions

2.46. For statements P and Q, show that $P \Rightarrow (P \vee Q)$ is a tautology.

2.47. For statements P and Q, show that $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ is a contradiction.

2.48. For statements P and Q, show that $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology. Then state $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ in words. (This is an important logical argument form, called **modus ponens**.)

2.49. For statements P, Q and R, show that $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology. Then state this compound statement in words. (This is another important logical argument form, called **syllogism**.)

2.50. Let R and S be compound statements involving the same component statements. If R is a tautology and S is a contradiction, then what can be said of the following?
(a) $R \vee S$ (b) $R \wedge S$ (c) $R \Rightarrow S$ (d) $S \Rightarrow R$.

Section 2.8: Logical Equivalence

2.51. For statements P and Q, the implication $(\sim P) \Rightarrow (\sim Q)$ is called the **inverse** of the implication $P \Rightarrow Q$.

(a) Use a truth table to show that these statements are not logically equivalent.
(b) Find another implication that is logically equivalent to $(\sim P) \Rightarrow (\sim Q)$ and verify your answer.

2.52. Let P and Q be statements.

(a) Is $\sim (P \vee Q)$ logically equivalent to $(\sim P) \vee (\sim Q)$? Explain.
(b) What can you say about the biconditional $\sim (P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$?

2.53. For statements P, Q and R, use a truth table to show that each of the following pairs of statements is logically equivalent.

(a) $(P \wedge Q) \Leftrightarrow P$ and $P \Rightarrow Q$.
(b) $P \Rightarrow (Q \vee R)$ and $(\sim Q) \Rightarrow ((\sim P) \vee R)$.

2.54. For statements P and Q, show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are logically equivalent.

2.55. For statements P, Q and R, show that $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent.

2.56. Two compound statements S and T are composed of the same component statements P, Q and R. If S and T are not logically equivalent, then what can we conclude from this?

2.57. Five compound statements S_1, S_2, S_3, S_4 and S_5 are all composed of the same component statements P and Q and whose truth tables have identical first and fourth rows. Show that at least two of these five statements are logically equivalent.

Section 2.9: Some Fundamental Properties of Logical Equivalence

2.58. Verify the following laws stated in Theorem 2.18:

(a) Let P, Q and R be statements. Then

$$P \vee (Q \wedge R) \text{ and } (P \vee Q) \wedge (P \vee R) \text{ are logically equivalent.}$$

(b) Let P and Q be statements. Then

$$\sim(P \vee Q) \text{ and } (\sim P) \wedge (\sim Q) \text{ are logically equivalent.}$$

2.59. Write negations of the following open sentences:

(a) Either $x = 0$ or $y = 0$.
(b) The integers a and b are both even.

2.60. Consider the implication: If x and y are even, then xy is even.

(a) State the implication using "only if."
(b) State the converse of the implication.
(c) State the implication as a disjunction (see Theorem 2.17).
(d) State the negation of the implication as a conjunction (see Theorem 2.21(a)).

2.61. For a real number x, let $P(x) : x^2 = 2$. and $Q(x) : x = \sqrt{2}$. State the negation of the biconditional $P \Leftrightarrow Q$ in words (see Theorem 2.21(b)).

2.62. Let P and Q be statements. Show that $[(P \vee Q) \wedge \sim (P \wedge Q)] \equiv \sim (P \Leftrightarrow Q)$.

2.63. Let $n \in \mathbf{Z}$. For which implication is its negation the following?
The integer $3n + 4$ is odd and $5n - 6$ is even.

2.64. For which biconditional is its negation the following?
n^3 and $7n + 2$ are odd or n^3 and $7n + 2$ are even.

Section 2.10: Quantified Statements

2.65. Let S denote the set of odd integers and let

$$P(x) : x^2 + 1 \text{ is even.} \quad \text{and} \quad Q(x) : x^2 \text{ is even.}$$

be open sentences over the domain S. State $\forall x \in S, P(x)$ and $\exists x \in S, Q(x)$ in words.

2.66. Define an open sentence $R(x)$ over some domain S and then state $\forall x \in S, R(x)$ and $\exists x \in S, R(x)$ in words.

2.67. State the negations of the following quantified statements, where all sets are subsets of some universal set U:

(a) For every set A, $A \cap \overline{A} = \emptyset$.
(b) There exists a set A such that $\overline{A} \subseteq A$.

2.68. State the negations of the following quantified statements:

(a) For every rational number r, the number $1/r$ is rational.
(b) There exists a rational number r such that $r^2 = 2$.

2.69. Let $P(n)$: $(5n-6)/3$ is an integer. be an open sentence over the domain $\mathbf{Z}$. Determine, with explanations, whether the following statements are true:
(a) $\forall n \in \mathbf{Z}, P(n)$. (b) $\exists n \in \mathbf{Z}, P(n)$.

2.70. Determine the truth value of each of the following statements.
(a) $\exists x \in \mathbf{R}, x^2 - x = 0$. (b) $\forall n \in \mathbf{N}, n+1 \geq 2$.
(c) $\forall x \in \mathbf{R}, \sqrt{x^2} = x$. (d) $\exists x \in \mathbf{Q}, 3x^2 - 27 = 0$.
(e) $\exists x \in \mathbf{R}, \exists y \in \mathbf{R}, x + y + 3 = 8$. (f) $\forall x, y \in \mathbf{R}, x + y + 3 = 8$.
(g) $\exists x, y \in \mathbf{R}, x^2 + y^2 = 9$. (h) $\forall x \in \mathbf{R}, \forall y \in \mathbf{R}, x^2 + y^2 = 9$.

2.71. The statement

For every integer m, either $m \leq 1$ or $m^2 \geq 4$.

can be expressed using a quantifier as:

$$\forall m \in \mathbf{Z}, m \leq 1 \text{ or } m^2 \geq 4.$$

Do this for the following two statements.

(a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.
(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.
(c) Express in words the negations of the statements in (a) and (b).
(d) Using quantifiers, express in symbols the negations of the statements in both (a) and (b).

2.72. Let $P(x)$ and $Q(x)$ be open sentences where the domain of the variable x is S. Which of the following implies that $(\sim P(x)) \Rightarrow Q(x)$ is false for some $x \in S$?

(a) $P(x) \wedge Q(x)$ is false for all $x \in S$.
(b) $P(x)$ is true for all $x \in S$.
(c) $Q(x)$ is true for all $x \in S$.
(d) $P(x) \vee Q(x)$ is false for some $x \in S$.
(e) $P(x) \wedge (\sim Q(x))$ is false for all $x \in S$.

2.73. Let $P(x)$ and $Q(x)$ be open sentences where the domain of the variable x is T. Which of the following implies that $P(x) \Rightarrow Q(x)$ is true for all $x \in T$?

(a) $P(x) \wedge Q(x)$ is false for all $x \in T$.
(b) $Q(x)$ is true for all $x \in T$.
(c) $P(x)$ is false for all $x \in T$.
(d) $P(x) \wedge (\sim (Q(x))$ is true for some $x \in T$.
(e) $P(x)$ is true for all $x \in T$.
(f) $(\sim P(x)) \wedge (\sim Q(x))$ is false for all $x \in T$.

2.74. Consider the open sentence

$$P(x, y, z): \ (x-1)^2 + (y-2)^2 + (z-2)^2 > 0.$$

where the domain of each of the variables x, y and z is $\mathbf{R}$.

(a) Express the quantified statement $\forall x \in \mathbf{R}, \forall y \in \mathbf{R}, \forall z \in \mathbf{R}, P(x, y, z)$ in words.
(b) Is the quantified statement in (a) true or false? Explain.
(c) Express the negation of the quantified statement in (a) in symbols.
(d) Express the negation of the quantified statement in (a) in words.
(e) Is the negation of the quantified statement in (a) true or false? Explain.

2.75. Consider quantified statement

For every $s \in S$ and $t \in S$, $st - 2$ is prime.

where the domain of the variables s and t is $S = \{3, 5, 11\}$.

(a) Express this quantified statement in symbols.
(b) Is the quantified statement in (a) true or false? Explain.
(c) Express the negation of the quantified statement in (a) in symbols.
(d) Express the negation of the quantified statement in (a) in words.
(e) Is the negation of the quantified statement in (a) true or false? Explain.

2.76. Let A be the set of circles in the plane with center (0, 0) and let B be the set of circles in the plane with center (1, 1). Furthermore, let

$$P(C_1, C_2)\text{: } C_1 \text{ and } C_2 \text{ have exactly two points in common.}$$

be an open sentence where the domain of C_1 is A and the domain of C_2 is B.

(a) Express the following quantified statement in words:

$$\forall C_1 \in A, \exists C_2 \in B, P(C_1, C_2). \quad \textbf{(2.30)}$$

(b) Express the negation of the quantified statement in (2.30) in symbols.
(c) Express the negation of the quantified statement in (2.30) in words.

2.77. For a triangle T, let $r(T)$ denote the ratio of the length of the longest side of T to the length of the smallest side of T. Let A denote the set of all triangles and let

$$P(T_1, T_2)\text{: } r(T_2) \geq r(T_1).$$

be an open sentence where the domain of both T_1 and T_2 is A.

(a) Express the following quantified statement in words:

$$\exists T_1 \in A, \forall T_2 \in A, P(T_1, T_2). \quad \textbf{(2.31)}$$

(b) Express the negation of the quantified statement in (2.31) in symbols.
(c) Express the negation of the quantified statement in (2.31) in words.

2.78. Consider the open sentence $P(a, b)$: $a/b < 1$. where the domain of a is $A = \{2, 3, 5\}$ and the domain of b is $B = \{2, 4, 6\}$.

(a) State the quantified statement $\forall a \in A, \exists b \in B, P(a, b)$ in words.
(b) Show the quantified statement in (a) is true.

2.79. Consider the open sentence $Q(a, b)$: $a - b < 0$. where the domain of a is $A = \{3, 5, 8\}$ and the domain of b is $B = \{3, 6, 10\}$.

(a) State the quantified statement $\exists b \in B, \forall a \in A, Q(a, b)$ in words.
(b) Show the quantified statement in (a) is true.

Section 2.11: Characterizations of Statements

2.80. Give a definition of each of the following and then state a characterization of each.

(a) Two lines in the plane are perpendicular.
(b) A rational number.

2.81. Define an integer n to be odd if n is not even. State a characterization of odd integers.

2.82. Define a triangle to be isosceles if it has two equal sides. Which of the following statements are characterizations of isosceles triangles? If a statement is not a characterization of isosceles triangles, then explain why.

(a) If a triangle is equilateral, then it is isosceles.
(b) A triangle T is isosceles if and only if T has two equal sides.
(c) If a triangle has two equal sides, then it is isosceles.
(d) A triangle T is isosceles if and only if T is equilateral.
(e) If a triangle has two equal angles, then it is isosceles.
(f) A triangle T is isosceles if and only if T has two equal angles.

2.83. By definition, a right triangle is a triangle one of whose angles is a right angle. Also, two angles in a triangle are complementary if the sum of their degrees is 90°. Which of the following statements are characterizations of a right triangle? If a statement is not a characterization of a right triangle, then explain why.

(a) A triangle is a right triangle if and only if two of its sides are perpendicular.
(b) A triangle is a right triangle if and only if it has two complementary angles.
(c) A triangle is a right triangle if and only if its area is half of the product of the lengths of some pair of its sides.
(d) A triangle is a right triangle if and only if the square of the length of its longest side equals to the sum of the squares of the lengths of the two smallest sides.
(e) A triangle is a right triangle if and only if twice of the area of the triangle equals the area of some rectangle.

2.84. Two distinct lines in the plane are defined to be parallel if they don't intersect. Which of the following is a characterization of parallel lines?

(a) Two distinct lines ℓ_1 and ℓ_2 are parallel if and only if any line ℓ_3 that is perpendicular to ℓ_1 is also perpendicular to ℓ_2.
(b) Two distinct lines ℓ_1 and ℓ_2 are parallel if and only if any line distinct from ℓ_1 and ℓ_2 that doesn't intersect ℓ_1 also doesn't intersect ℓ_2.
(c) Two distinct lines ℓ_1 and ℓ_2 are parallel if and only if whenever a line ℓ intersects ℓ_1 in an acute angle α, then ℓ also intersects ℓ_2 in an acute angle α.
(d) Two distinct lines ℓ_1 and ℓ_2 are parallel if and only if whenever a point P is not on ℓ_1, the point P is not on ℓ_2.

ADDITIONAL EXERCISES FOR CHAPTER 2

2.85. Construct a truth table for $P \wedge (Q \Rightarrow (\sim P))$.

2.86. Given that the implication $(Q \vee R) \Rightarrow (\sim P)$ is false and Q is false, determine the truth values of R and P.

2.87. Find a compound statement involving the component statements P and Q that has the truth table given in Figure 2.18.

P	Q	$\sim Q$	
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Figure 2.18 Truth table for Exercise 2.87.

2.88. Determine the truth value of each of the following quantified statements:
(a) $\exists x \in \mathbf{R}, x^3 + 2 = 0.$ (b) $\forall n \in \mathbf{N}, 2 \geq 3 - n.$
(c) $\forall x \in \mathbf{R}, |x| = x.$ (d) $\exists x \in \mathbf{Q}, x^4 - 4 = 0.$
(e) $\exists x, y \in \mathbf{R}, x + y = \pi.$ (f) $\forall x, y \in \mathbf{R}, x + y = \sqrt{x^2 + y^2}.$

2.89. Rewrite each of the implications below using (1) only if and (2) sufficient.

(a) If a function f is differentiable, then f is continuous.
(b) If $x = -5$, then $x^2 = 25$.

2.90. Let $P(n)$: $n^2 - n + 5$ is a prime. be an open sentence over a domain S.

(a) Determine the truth values of the quantified statements $\forall n \in S, P(n)$ and $\exists n \in S, \sim P(n)$ for $S = \{1, 2, 3, 4\}$.
(b) Determine the truth values of the quantified statements $\forall n \in S, P(n)$ and $\exists n \in S, \sim P(n)$ for $S = \{1, 2, 3, 4, 5\}$.
(c) How are the statements in (a) and (b) related?

2.91. (a) For statements P, Q and R, show that

$$((P \wedge Q) \Rightarrow R) \equiv ((P \wedge (\sim R)) \Rightarrow (\sim Q)).$$

(b) For statements P, Q and R, show that

$$((P \wedge Q) \Rightarrow R) \equiv ((Q \wedge (\sim R)) \Rightarrow (\sim P)).$$

2.92. For a fixed integer n, use Exercise 2.91 to restate the following implication in two different ways:

If n is a prime and $n > 2$, then n is odd.

2.93. For fixed integers m and n, use Exercise 2.91 to restate the following implication in two different ways:

If m is even and n is odd, then $m + n$ is odd.

2.94. For a real-valued function f and a real number x, use Exercise 2.91 to restate the following implication in two different ways:

If $f'(x) = 3x^2 - 2x$ and $f(0) = 4$, then $f(x) = x^3 - x^2 + 4$.

2.95. For the set $S = \{1, 2, 3\}$, give an example of three open sentences $P(n)$, $Q(n)$ and $R(n)$, each over the domain S, such that (1) each of $P(n)$, $Q(n)$ and $R(n)$ is a true statement for exactly two elements of S, (2) all of the implications $P(1) \Rightarrow Q(1)$, $Q(2) \Rightarrow R(2)$ and $R(3) \Rightarrow P(3)$ are true, and (3) the converse of each implication in (2) is false.

2.96. Do there exist a set S of cardinality 2 and a set $\{P(n), Q(n), R(n)\}$ of three open sentences over the domain S such that (1) the implications $P(a) \Rightarrow Q(a)$, $Q(b) \Rightarrow R(b)$ and $R(c) \Rightarrow P(c)$ are true, where $a, b, c \in S$ and (2) the converses of the implications in (1) are false? Necessarily, at least two of these elements a, b and c of S are equal.

2.97. Let $A = \{1, 2, \ldots, 6\}$ and $B = \{1, 2, \ldots, 7\}$. For $x \in A$, let $P(x) : 7x + 4$ is odd. For $y \in B$, let $Q(y) : 5y + 9$ is odd. Let

$$S = \{(P(x), Q(y)) : x \in A, y \in B, P(x) \Rightarrow Q(y) \text{ is false}\}.$$

What is $|S|$?

2.98. Let $P(x, y, z)$ be an open sentence, where the domains of x, y and z are A, B and C, respectively.

(a) State the quantified statement $\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z)$ in words.
(b) State the quantified statement $\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z)$ in words for $P(x, y, z) : x = yz$.

(c) Determine whether the quantified statement in (b) is true when $A = \{4, 8\}$, $B = \{2, 4\}$ and $C = \{1, 2, 4\}$.

2.99. Let $P(x, y, z)$ be an open sentence, where the domains of x, y and z are A, B and C, respectively.

(a) Express the negation of $\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z)$ in symbols.
(b) Express $\sim (\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z))$ in words.
(c) Determine whether $\sim (\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z))$ is true when $P(x, y, z): \ x + z = y$. for $A = \{1, 3\}$, $B = \{3, 5, 7\}$ and $C = \{0, 2, 4, 6\}$.

2.100. Write each of the following using "if, then."

(a) A sufficient condition for a triangle to be isosceles is that it has two equal angles.
(b) Let C be a circle of diameter $\sqrt{2/\pi}$. Then the area of C is $1/2$.
(c) The 4th power of every odd integer is odd.
(d) Suppose that the slope of a line ℓ is 2. Then the equation of ℓ is $y = 2x + b$ for some real number b.
(e) Whenever a and b are nonzero rational numbers, a/b is a nonzero rational number.
(f) For every three integers, there exist two of them whose sum is even.
(g) A triangle is a right triangle if the sum of two of its angles is 90°.
(h) The number $\sqrt{3}$ is irrational.